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Generalization of $(\in, \in \vee q_k)$-fuzzy subnear-rings and $(\in, \in \vee q_k)$-fuzzy ideals of near-rings

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Abstract. In this paper, we introduce the notions of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring and $(\in, \in \vee q_\delta^k)$-fuzzy ideal of a near-ring which are generalization of $(\in, \in \vee q_k)$-fuzzy subnear-ring and $(\in, \in \vee q_k)$-fuzzy ideal respectively. We provide the characterizations of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of a near-ring and deal with some related properties. Further, we introduce the notion of $\in \vee q_\delta^k$-level subset of a fuzzy subset and obtain some related results.

1. Introduction

The basic concepts of fuzzy sets and their properties was proposed by Zadeh in [28]. Extensive applications of fuzzy set theory have been found in various fields such as control engineering, computer science, expert systems, artificial intelligence, operations research, management science, robotics, pattern recognition and others. Rosenfeld [24] initiated the notion of fuzzy subgroup of a group. Since then, the study of fuzzy algebraic structures have been carried out in many directions such as semigroups, groups, near-rings, rings, modules, vector spaces, topology and so on. The notion of fuzzy point and its "belongingness" and "quasi-coincidence with" a fuzzy subset, which was introduced by Pu and Liu [23], played a vital role to generate different types of fuzzy subgroups. It is important to mention that Bhakat and Das [3, 4] defined the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincidence with" relation (q) between a fuzzy point and a fuzzy subset, and introduced $(\in, \in \vee q)$-fuzzy subgroup which is an useful generalization of Rosenfeld's fuzzy subgroup. In [5], they defined the notions of $(\in, \in \vee q)$-fuzzy subring and $(\in, \in \vee q)$-fuzzy ideal of a ring. Abou-Zaid [1] initiated the notions of fuzzy subnear-ring and fuzzy ideal of a near-ring. In [19], Kim and Kim studied some properties of fuzzy ideals of near-rings. Saikia and Bharthakur [25] discussed the notions of fuzzy N-subgroups and fuzzy ideals of near-rings and near-ring groups. Narayanan and Manikantan [21] introduced the notions of $(\in, \in \vee q)$-fuzzy subnear-ring, $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q)$-fuzzy quasi ideal of a near-ring. In [26], Shabir et al. characterized the regular semigroups in terms of $(\in, \in \vee q)$-fuzzy left (resp. right, quasi, bi-) ideals. Dheena and Coumaressane [6] introduced the concept of $(\in, \in \vee q_k)$-fuzzy subnear-ring (resp. ideal) of a near-ring which is a generalization of $(\in, \in \vee q)$-fuzzy subnear-ring (resp. ideal). Jun [9] defined the notion of $(\in, \in \vee q_k)$-fuzzy subalgebra in BCK/BCI-algebras and investigated several properties. In [27], Shabir et al. introduced the notion of

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$(\in, \in \vee q_k)$-fuzzy left (resp. right, quasi, bi-) ideal of a semigroup. Jun et al. [10] introduced the concept of $(\in, \in \vee q_k)$-fuzzy (normal) subgroup of a group which is a generalization of $(\in, \in \vee q)$-fuzzy (normal) subgroup. Faisal et al. [7] gave the characterizations of Γ-semigroups in terms of $(\in, \in \vee q_k)$-fuzzy Γ-left (resp. right, two-sided, bi-, quasi-, interior) ideals. In [8], Gulistan et al. defined the notion of $(\in, \in \vee q_k)$-fuzzy KU-ideal of KU-algebra and provided some related results. Khan et al. [17] studied the concept of an $(\in, \in \vee q_k)$-intuitionistic fuzzy (soft) ideal of subtraction algebras. Azhar et al. [2] introduced the concept of $(\in, \in \vee q_k)$-fuzzy hyperideal of an ordered LA-semihypergroup. In [12], Jun et al. considered the generalized form of quasi-coincident with relation (q) and introduced the concept of $(\in, \in \vee q_0^\delta)$-fuzzy subgroup of a group and discussed related results. Jun and Song [14] generalized the concepts and results in [26], and provided the characterizations of $(\in, \in \vee q_0^\delta)$-fuzzy left (resp. right, bi-) ideals of semigroups. They also discussed the characterizations of $(\in, \in \vee q_0^\delta)$-fuzzy generalized bi-ideals in semigroups [15]. In [13], Jun et al. provided the characterizations of $(\in, \in \vee q_0^\delta)$-fuzzy subsemigroups of semigroups. In [11], Jun and Öztürk introduced the notions of $(\in, \in \vee q_0^\delta)$-fuzzy subrings and $(\in, \in \vee q_0^\delta)$-fuzzy ideals of a ring and obtained some properties. Kang [16] introduced the notion of $(\in, \in \vee q_\delta^k)$-fuzzy subsemigroup in a semigroup and investigated some of its properties. Khan et al. [18] defined the concept of $(\in, \in \vee q_\delta^k)$-fuzzy left (resp. right, quasi, bi-) ideal in a semigroup and studied some related properties. Motivated by the above concepts, in this paper, we introduce the notions of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring and $(\in, \in \vee q_\delta^k)$-fuzzy ideal of a near-ring which are generalization of $(\in, \in \vee q_k)$-fuzzy subnear-ring and $(\in, \in \vee q_k)$-fuzzy ideal respectively. We provide the characterizations of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of a near-ring and deal with some related properties. Further, we introduce the notion of $\in \vee q_\delta^k$-level subset of a fuzzy subset and obtain some related results.

2. Preliminaries

We first recall some basic definitions and results proposed by the early pioneers.

By a *near-ring* [22] we mean a non-empty set N with two binary operations " + " and " · " satisfying the following axioms:

(i) $(N, +)$ is a group (not necessarily an abelian),

(ii) $(N, \cdot)$ is a semigroup,

(iii) $(x + y) \cdot z = x \cdot z + y \cdot z$ for all $x, y, z \in N$.

Precisely speaking, it is a right near-ring because it satisfies the right distributive law. We will use the word near-ring to mean right near-ring. We denote xy instead of $x \cdot y$. Note that $0x = 0$ for all $x \in N$, but in general $x0 \neq 0$ for some $x \in N$.

Let N be a near-ring. If A and B are two non-empty subsets of N, we define $AB = \{ab | a \in A, b \in B\}$ and $A * B = \{a(b + i) - ab | a, b \in A, i \in B\}$.

A subgroup A of a near-ring N is called a *subnear-ring* of N if $AA \subseteq A$. A subset I of a near-ring N is called an *ideal* of N if (i) $(I, +)$ is a normal subgroup of $(N, +)$, (ii) $IN \subseteq I$, (iii) $x(y + i) - xy \in I$ for all $x, y \in N$ and $i \in I$, that is, $N * I \subseteq I$.

A normal subgroup R of $(N, +)$ with (ii) is called a *right ideal* of N while a normal subgroup L of $(N, +)$ with (iii) is called a *left ideal* of N.

A *fuzzy subset* λ of a non-empty set X is a mapping from X to $[0, 1]$(see [28]).

Let λ and μ be two fuzzy subset of a semigroup X. We define the relation $\subseteq$ between λ and μ, the intersection, the union and product of λ and μ, respectively as follows:

(i) $\lambda \subseteq \mu$ if $\lambda(x) \leq \mu(x)$ for all $x \in X$, (ii) $(\lambda \cap \mu)(x) = \min\{\lambda(x), \mu(x)\}$ for all $x \in X$, (iii) $(\lambda \cup \mu)(x) = \max\{\lambda(x), \mu(x)\}$ for all $x \in X$ and (iv) $(\lambda \circ \mu)(x) = \begin{cases} \sup\limits_{x=yz}\{\min\{\lambda(y), \mu(z)\}\} & \text{if } x = yz \text{ and } y, z \in X, \\ 0 & \text{otherwise,} \end{cases}$

for every $x \in X$. It is easily verified that the "product" of fuzzy subsets is associative.

Let λ be a fuzzy subset of a set X and $t \in [0,1]$. Then the set $\lambda_t = \{x \in X | \lambda(x) \geq t\}$ is called a *level subset* of λ in X (see [1]). A fuzzy subset λ of a set X is of the form

$$\lambda(y) = \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t (see [23]). For a fuzzy point x_t and a fuzzy subset λ in a set X, we say that

(i) $x_t \in \lambda$ if $\lambda(x) \geq t$. In this case, x_t is said to *belong* to a fuzzy subset λ (see [23]).
(ii) $x_t q \lambda$ if $\lambda(x) + t > 1$. In this case, x_t is said to be *quasi-coincident* with a fuzzy subset λ (see [23]).
(iii) $x_t q_k \lambda$ if $\lambda(x) + t > 1 - k$, where $k \in [0,1)$. In this case, x_t is said to be *k-quasi-coincident* with a fuzzy subset λ (see [6]).
(iv) $x_t q_\delta \lambda$ if $\lambda(x) + t > \delta$, where $\delta \in (0,1]$. In this case, x_t is said to be *δ-quasi-coincident* with a fuzzy subset λ (see [12]).
(v) $x_t q_\delta^k \lambda$ if $\lambda(x) + t > \delta - k$, where $k, \delta \in [0,1]$ such that $k < \delta$. In this case, x_t is said to be *(δ, k)-quasi-coincident* with a fuzzy subset λ (see [16]).

To say that $x_t \in \vee q_\delta^k \lambda$ (resp. $x_t \in \wedge q_\delta^k \lambda$) means that $x_t \in \lambda$ or $x_t q_\delta^k \lambda$ (resp. $x_t \in \lambda$ and $x_t q_\delta^k \lambda$) and $x_t \overline{\in} \lambda$, $x_t \overline{\in \vee q_\delta^k} \lambda$ will respectively mean $x_t \in \lambda$ and $x_t \in \vee q_\delta^k \lambda$ does not hold.

Definition 2.1. *[1] A fuzzy subset λ of a near-ring N is called a fuzzy subnear-ring of N if for all $x, y \in N$,*

(i) $\lambda(x - y) \geq \min\{\lambda(x), \lambda(y)\}$,
(ii) $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$.

Definition 2.2. *[1] A fuzzy subset λ of a near-ring N is called a fuzzy ideal of N if it satisfies:*

(i) *λ is a fuzzy subnear-ring of N,*
(ii) *$\lambda(y + x - y) \geq \lambda(x)$ for all $x, y \in N$,*
(iii) *$\lambda(xy) \geq \lambda(x)$ for all $x, y \in N$,*
(iv) *$\lambda(x(y + z) - xy) \geq \lambda(z)$ for all $x, y, z \in N$.*

A fuzzy subset with (i), (ii) and (iii) is called a fuzzy right ideal of N whereas a fuzzy subset with (i), (ii) and (iv) is called a fuzzy left ideal of N.

Definition 2.3. *[21] A fuzzy subset λ of a near-ring N is called an $(\in, \in \vee q)$-fuzzy subnear-ring of N if for all $x, y \in N$ and $s, t \in (0,1]$,*

(i) *$x_s \in \lambda$ and $y_t \in \lambda$ implies $(x + y)_{\min\{s,t\}} \in \vee q \lambda$,*
(ii) *$x_s \in \lambda$ implies $(-x)_s \in \vee q \lambda$,*
(iii) *$x_s \in \lambda$ and $y_t \in \lambda$ implies $(xy)_{\min\{s,t\}} \in \vee q \lambda$.*

Definition 2.4. *[21] A fuzzy subset λ of a near-ring N is called an $(\in, \in \vee q)$-fuzzy ideal of N if*

(i) *λ is an $(\in, \in \vee q)$-fuzzy subnear-ring of N,*
(ii) *$x_s \in \lambda$ and $y \in N$ implies $(y + x - y)_s \in \vee q \lambda$ for all $x, y \in N$ and $s \in (0,1]$,*
(iii) *$x_s \in \lambda$ and $y \in N$ implies $(xy)_s \in \vee q \lambda$ for all $x, y \in N$ and $s \in (0,1]$,*
(iv) *$z_s \in \lambda$ and $x, y \in N$ implies $(x(y + z) - xy)_s \in \vee q \lambda$ for all $x, y, z \in N$ and $s \in (0,1]$.*

Definition 2.5. *[6] A fuzzy subset λ of a near-ring N is called an $(\in, \in \vee q_k)$-fuzzy subnear-ring of N if for all $x, y \in N$, $s, t \in (0,1]$ and $k \in [0,1)$,*

(i) *$x_s \in \lambda$ and $y_t \in \lambda$ implies $(x + y)_{\min\{s,t\}} \in \vee q_k \lambda$,*
(ii) *$x_s \in \lambda$ implies $(-x)_s \in \vee q_k \lambda$,*
(iii) *$x_s \in \lambda$ and $y_t \in \lambda$ implies $(xy)_{\min\{s,t\}} \in \vee q_k \lambda$.*

Definition 2.6. *[6] A fuzzy subset λ of a near-ring N is called an $(\in, \in \vee q_k)$-fuzzy ideal of N if for all $x, y, z \in N$ and $s, t \in (0, 1]$ and $k \in [0, 1)$,*

(i) *$x_s \in \lambda$ and $y_t \in \lambda$ implies $(x - y)_{\min\{s,t\}} \in \vee q_k \lambda$,*
(ii) *$x_s \in \lambda$ and $y \in N$ implies $(y + x - y)_s \in \vee q_k \lambda$,*
(iii) *$x_s \in \lambda$ and $y \in N$ implies $(xy)_s \in \vee q_k \lambda$,*
(iv) *$z_s \in \lambda$ and $x, y \in N$ implies $(x(y + z) - xy)_s \in \vee q_k \lambda$.*

Throughout this paper, N will denote a right near-ring and $0 \leq k < \delta \leq 1$ unless otherwise specified. We denote by χ_A the characteristic function of a subset A of N.

3. $(\in, \in \vee q^k_\delta)$-fuzzy subnear-rings and $(\in, \in \vee q^k_\delta)$-fuzzy ideals

In this section, we introduce the notion of $(\in, \in \vee q^k_\delta)$-fuzzy subgroup of a group. And, we introduce the notions of $(\in, \in \vee q^k_\delta)$-fuzzy subnear-ring and $(\in, \in \vee q^k_\delta)$-fuzzy ideal of a near-ring which are generalization of $(\in, \in \vee q_k)$-fuzzy subnear-ring and $(\in, \in \vee q_k)$-fuzzy ideal respectively.

Definition 3.1. *A fuzzy subset λ of a group $(N, +)$ is said to be an $(\in, \in \vee q^k_\delta)$-fuzzy subgroup of N if for all $x, y \in N$ and $s, t \in (0, 1]$,*

$(a1)$ *$x_s \in \lambda$ and $y_t \in \lambda$ implies $(x + y)_{\min\{s,t\}} \in \vee q^k_\delta \lambda$,*
$(a2)$ *$x_s \in \lambda$ implies $(-x)_s \in \vee q^k_\delta \lambda$.*

Equivalently:
$(a3)$ *$x_s \in \lambda$ and $y_t \in \lambda$ implies $(x - y)_{\min\{s,t\}} \in \vee q^k_\delta \lambda$.*

Lemma 3.2. *The conditions $(a1), (a2)$ and $(a3)$ in Definition 3.1 are equivalent to*

$(a1')$ *$\lambda(x + y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in N$,*
$(a2')$ *$\lambda(-x) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$ for all $x \in N$,*
$(a3')$ *$\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in N$,*

respectively.

Proof. $(a1) \Rightarrow (a1')$: Let $x, y \in N$ and $t \in (0, 1]$ such that $x_t \in \lambda$ and $y_t \in \lambda$. Then $(x + y)_t \in \vee q^k_\delta \lambda$. Suppose that $\lambda(x + y) < \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Choose t such that $\lambda(x + y) < t \leq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Then $\lambda(x) \geq t$, $\lambda(y) \geq t$ and $\lambda(x + y) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\lambda(x + y) + t < 2t \leq \delta - k$. That is, $(x + y)_t \overline{\in} \lambda$ and $(x + y)_t \overline{q^k_\delta} \lambda$. This implies that $(x + y)_t \overline{\in \vee q^k_\delta} \lambda$, which is a contradiction. Hence $\lambda(x + y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$.

$(a1') \Rightarrow (a1)$: Let $x, y \in N$ and $s, t \in (0, 1]$ be such that $x_s \in \lambda$, $y_t \in \lambda$ but $(x + y)_{\min\{s,t\}} \overline{\in \vee q^k_\delta} \lambda$. Then $\lambda(x) \geq s$, $\lambda(y) \geq t$, $\lambda(x + y) < \min\{s, t\}$ and $\lambda(x + y) + \min\{s, t\} \leq \delta - k$. This implies that $\lambda(x + y) < \min\{\lambda(x), \lambda(y)\}$ and $\lambda(x + y) < \frac{\delta-k}{2}$. Thus $\lambda(x + y) < \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_s \in \lambda$, $y_t \in \lambda$ implies $(x + y)_{\min\{s,t\}} \in \vee q^k_\delta \lambda$.

$(a2) \Rightarrow (a2')$: Let $x \in N$ and $s \in (0, 1]$ such that $x_s \in \lambda$. Then $(-x)_s \in \vee q^k_\delta \lambda$. Suppose that $\lambda(-x) < \min\{\lambda(x), \frac{\delta-k}{2}\}$. Choose s such that $\lambda(-x) < s \leq \min\{\lambda(x), \frac{\delta-k}{2}\}$. Then $\lambda(x) \geq s$ and $\lambda(-x) < s$. Since $s \leq \frac{\delta-k}{2}$, we have $\lambda(-x) + s < 2s \leq \delta - k$. That is, $(-x)_s \overline{\in} \lambda$ and $(-x)_s \overline{q^k_\delta} \lambda$. It follows that $(-x)_s \overline{\in \vee q^k_\delta} \lambda$, which is a contradiction. Therefore $\lambda(-x) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$.

$(a2') \Rightarrow (a2)$: Let $x \in N$ and $s \in (0, 1]$ be such that $x_s \in \lambda$, but $(-x)_s \overline{\in \vee q^k_\delta} \lambda$. Then $\lambda(x) \geq s$, $\lambda(-x) < s$ and $\lambda(-x) + s \leq \delta - k$. This implies that $\lambda(-x) < \lambda(x)$ and $\lambda(-x) < \frac{\delta-k}{2}$. Thus $\lambda(-x) < \min\{\lambda(x), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_s \in \lambda$ implies $(-x)_s \in \vee q^k_\delta \lambda$.

$(a3) \Rightarrow (a3')$: Let $x, y \in N$ and $t \in (0, 1]$ such that $x_t \in \lambda$ and $y_t \in \lambda$. Then $(x - y)_t \in \vee q^k_\delta \lambda$. Suppose that $\lambda(x - y) < \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Choose t such that $\lambda(x - y) < t \leq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Then $\lambda(x) \geq t$, $\lambda(y) \geq t$

and $\lambda(x-y) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\lambda(x-y)+t < 2t \leq \delta-k$. That is, $(x-y)_t \overline{\in} \lambda$ and $(x-y)_t \overline{q_\delta^k} \lambda$. This implies that $(x-y)_t \overline{\in \vee q_\delta^k} \lambda$, which is a contradiction. Hence $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$.

$(a3') \Rightarrow (a3)$: Let $x, y \in N$ and $s, t \in (0,1]$ be such that $x_s \in \lambda$, $y_t \in \lambda$ but $(x-y)_{\min\{s,t\}} \overline{\in \vee q_\delta^k} \lambda$. Then $\lambda(x) \geq s$, $\lambda(y) \geq t$, $\lambda(x-y) < \min\{s,t\}$ and $\lambda(x-y) + \min\{s,t\} \leq \delta - k$. This implies that $\lambda(x-y) < \min\{\lambda(x), \lambda(y)\}$ and $\lambda(x-y) < \frac{\delta-k}{2}$. Thus $\lambda(x-y) < \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_s \in \lambda$, $y_t \in \lambda$ implies $(x-y)_{\min\{s,t\}} \in \vee q_\delta^k \lambda$. □

Remark 3.3. *The condition $(a1')$ with the condition $(a2')$ in Lemma 3.2 is equivalent to the condition: $(a3')$ $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in N$.*

Theorem 3.4. *A fuzzy subset λ of a group $(N,+)$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subgroup of N if and only if $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in N$.*

Proof. The proof follows from Lemma 3.2. □

If we take $\delta = 1$ and $k = 0$ in Theorem 3.4, we have the following corollary.

Corollary 3.5. ([4], *Remark 3.4) A fuzzy subset λ of a group N is an $(\in, \in \vee q)$-fuzzy subgroup of N if and only if $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ for all $x, y \in N$.*

If we take $\delta = 1$ in Theorem 3.4, we have the following corollary.

Corollary 3.6. ([6], *Theorem 3.4) A fuzzy subset λ of a group N is an $(\in, \in \vee q_k)$-fuzzy subgroup of N if and only if $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ for all $x, y \in N$.*

If we take $k = 0$ in Theorem 3.4, we have the following corollary.

Corollary 3.7. ([12], *Theorem 3.6) A fuzzy subset λ of a group N is an $(\in, \in \vee q_\delta)$-fuzzy subgroup of N if and only if for all $x, y \in G$, $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in N$.*

Definition 3.8. *A fuzzy subset λ of N is said to be an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N if for all $x, y \in N$ and $s, t \in (0,1]$,*

(b1) $x_s \in \lambda$ and $y_t \in \lambda$ implies $(x+y)_{\min\{s,t\}} \in \vee q_\delta^k \lambda$,
(b2) $x_s \in \lambda$ implies $(-x)_s \in \vee q_\delta^k \lambda$,
(b3) $x_s \in \lambda$ and $y_t \in \lambda$ implies $(xy)_{\min\{s,t\}} \in \vee q_\delta^k \lambda$.

An $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N with $k = 0$ is called an $(\in, \in \vee q_\delta)$-fuzzy subnear-ring of N .

Lemma 3.9. *The conditions $(b1)$, $(b2)$ and $(b3)$ in Definition 3.8 are equivalent to*

$(b1')$ $\lambda(x+y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in N$,
$(b2')$ $\lambda(-x) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$ for all $x \in N$,
$(b3')$ $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in N$,

respectively.

Proof. $(b1) \Leftrightarrow (b1')$ and $(b2) \Leftrightarrow (b2')$ follows from Lemma 3.2.

$(b3) \Rightarrow (b3')$: Let $x, y \in N$ and $t \in (0,1]$ such that $x_t \in \lambda$ and $y_t \in \lambda$. Then $(xy)_t \in \vee q_\delta^k \lambda$. Suppose that $\lambda(xy) < \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Choose t such that $\lambda(xy) < t \leq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Then $\lambda(x) \geq t$, $\lambda(y) \geq t$ and $\lambda(xy) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\lambda(xy) + t < 2t \leq \delta - k$. That is, $(xy)_t \overline{\in} \lambda$ and $(xy)_t \overline{q_\delta^k} \lambda$. This implies that $(xy)_t \overline{\in \vee q_\delta^k} \lambda$, which is a contradiction. Hence $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$.

$(b3') \Rightarrow (b3)$: Let $x, y \in N$ and $s, t \in (0,1]$ be such that $x_s \in \lambda$, $y_t \in \lambda$ but $(xy)_{\min\{s,t\}} \overline{\in \vee q_\delta^k} \lambda$. Then $\lambda(x) \geq s$, $\lambda(y) \geq t$, $\lambda(xy) < \min\{s,t\}$ and $\lambda(xy) + \min\{s,t\} \leq \delta - k$. This implies that $\lambda(xy) < \min\{\lambda(x), \lambda(y)\}$ and $\lambda(xy) < \frac{\delta-k}{2}$. Thus $\lambda(xy) < \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_s \in \lambda$, $y_t \in \lambda$ implies $(xy)_{\min\{s,t\}} \in \vee q_\delta^k \lambda$. □

Theorem 3.10. *A fuzzy subset λ of N is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N if and only if for all $x, y \in N$,*

(i) $\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$,
(ii) $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$.

Proof. The proof follows from Lemma 3.9. □

If we take $\delta = 1$ and $k = 0$ in Theorem 3.10, we have the following corollary.

Corollary 3.11. ([21], *Theorem 3.3) A fuzzy subset λ of N is an $(\in, \in \vee q)$-fuzzy subnear-ring of N if and only if $\lambda(x - y), \lambda(xy) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ for all $x, y \in N$.*

If we take $\delta = 1$ in Theorem 3.10, we have the following corollary.

Corollary 3.12. ([6], *Theorem 3.4) A fuzzy subset λ of N is an $(\in, \in \vee q_k)$-fuzzy subnear-ring of N if and only if $\lambda(x - y), \lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ for all $x, y \in N$.*

If we take $k = 0$ in Theorem 3.10, we have the following corollary.

Corollary 3.13. *A fuzzy subset λ of N is an $(\in, \in \vee q_\delta)$-fuzzy subnear-ring of N if and only if $\lambda(x - y), \lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in N$.*

Remark 3.14. *Every fuzzy subnear-ring, $(\in, \in \vee q)$-fuzzy subnear-ring, $(\in, \in \vee q_k)$-fuzzy subnear-ring and $(\in, \in \vee q_\delta)$-fuzzy subnear-ring of N is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N, but the converses are not necessarily true as shown in the following example.*

Example 3.15. *Let $N = \{0, a, b, c\}$ be the near-ring with $(N, +)$ as the Klein's four group and $(N, \cdot)$ as defined below (Scheme 19 : (7,7,7,1) See [22], p. 408).*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	a	a	a	c

Define a fuzzy subset λ of N by $\lambda(0) = 0.6$, $\lambda(a) = 0.3$, $\lambda(b) = 0.4$ and $\lambda(c) = 0.8$. Then λ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N with $\delta = 0.8$ and $k = 0.2$. But

(i) *λ is not a fuzzy subnear-ring of N, since* $0.3 = \lambda(a) = \lambda(b - c) \ngeq \min\{\lambda(b), \lambda(c)\} = \min\{0.4, 0.8\} = 0.4$.
(ii) λ is not an $(\in, \in \vee q)$-fuzzy subnear-ring of N, since $b_{0.35} \in \lambda$, $c_{0.45} \in \lambda$ and $(b + c)_{0.35} \overline{\in \vee q}\, \lambda$.
(iii) λ is not an $(\in, \in \vee q_k)$-fuzzy subnear-ring of N, since $b_{0.35} \in \lambda$, $c_{0.45} \in \lambda$ and $(b + c)_{0.35} \overline{\in \vee q_k}\, \lambda$.
(iv) λ is not an $(\in, \in \vee q_\delta)$-fuzzy subnear-ring of N, since $b_{0.35} \in \lambda$, $c_{0.45} \in \lambda$ and $(b + c)_{0.35} \overline{\in \vee q_\delta}\, \lambda$.

Definition 3.16. *A fuzzy subset λ of N is called an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N if for all $x, y, z \in N$ and $s, t \in (0, 1]$,*

$(c1)$ *$x_s \in \lambda$ and $y_t \in \lambda$ implies $(x - y)_{\min\{s,t\}} \in \vee q_\delta^k\, \lambda$,*
$(c2)$ *$x_s \in \lambda$ and $y \in N$ implies $(y + x - y)_s \in \vee q_\delta^k\, \lambda$,*
$(c3)$ *$x_s \in \lambda$ and $y \in N$ implies $(xy)_s \in \vee q_\delta^k\, \lambda$,*
$(c4)$ *$z_s \in \lambda$ and $x, y \in N$ implies $(x(y + z) - xy)_s \in \vee q_\delta^k\, \lambda$.*

A fuzzy subset λ with conditions $(c1)$, $(c2)$ and $(c3)$ is called an $(\in, \in \vee q_\delta^k)$-fuzzy right ideal of N while λ satisfies $(c1)$, $(c2)$ and $(c4)$ is called an $(\in, \in \vee q_\delta^k)$-fuzzy left ideal of N.

An $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N with $k = 0$ is called an $(\in, \in \vee q_\delta)$-fuzzy ideal of N.

Lemma 3.17. *The conditions $(c1)$ to $(c4)$ in Definition 3.16 are equivalent to*

($c1'$) $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ *for all* $x, y \in N$,
($c2'$) $\lambda(y+x-y) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$ *for all* $x, y \in N$,
($c3'$) $\lambda(xy) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$ *for all* $x, y \in N$,
($c4'$) $\lambda(x(y+z)-xy) \geq \min\{\lambda(z), \frac{\delta-k}{2}\}$ *for all* $x, y, z \in N$,

respectively.

Proof. ($c1$) $\Leftrightarrow$ ($c1'$) follows from Lemma 3.2.

($c2$) $\Rightarrow$ ($c2'$): Let $x, y \in N$ and $t \in (0,1]$ such that $x_t \in \lambda$. Then $(y+x-y)_t \in \vee q_\delta^k \lambda$. Suppose that $\lambda(y+x-y) < \min\{\lambda(x), \frac{\delta-k}{2}\}$. Choose t such that $\lambda(y+x-y) < t \leq \min\{\lambda(x), \frac{\delta-k}{2}\}$. Then $\lambda(x) \geq t$ and $\lambda(y+x-y) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\lambda(y+x-y)+t < 2t \leq \delta-k$. That is, $(y+x-y)_t \overline{\in} \lambda$ and $(y+x-y)_t \overline{q_\delta^k} \lambda$. This implies that $(y+x-y)_t \overline{\in \vee q_\delta^k} \lambda$, which is a contradiction. Hence $\lambda(y+x-y) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$.

($c2'$) $\Rightarrow$ ($c2$): Let $x, y \in N$ and $t \in (0,1]$ be such that $x_t \in \lambda$, but $(y+x-y)_t \overline{\in \vee q_\delta^k} \lambda$. Then $\lambda(x) \geq t$, $\lambda(y+x-y) < t$ and $\lambda(y+x-y)+t \leq \delta-k$. This implies that $\lambda(y+x-y) < \lambda(x)$ and $\lambda(y+x-y) < \frac{\delta-k}{2}$. Thus $\lambda(y+x-y) < \min\{\lambda(x), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_t \in \lambda$ and $y \in N$ implies $(y+x-y)_t \in \vee q_\delta^k \lambda$.

($c3$) $\Rightarrow$ ($c3'$): Let $x, y \in N$ and $t \in (0,1]$ such that $x_t \in \lambda$. Then $(xy)_t \in \vee q_\delta^k \lambda$. Suppose that $\lambda(xy) < \min\{\lambda(x), \frac{\delta-k}{2}\}$. Choose t such that $\lambda(xy) < t \leq \min\{\lambda(x), \frac{\delta-k}{2}\}$. Then $\lambda(x) \geq t$ and $\lambda(xy) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\lambda(xy)+t < 2t \leq \delta-k$. That is, $(xy)_t \overline{\in} \lambda$ and $(xy)_t \overline{q_\delta^k} \lambda$. This implies that $(xy)_t \overline{\in \vee q_\delta^k} \lambda$, which is a contradiction. Hence $\lambda(xy) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$.

($c3'$) $\Rightarrow$ ($c3$): Let $x, y \in N$ and $t \in (0,1]$ be such that $x_s \in \lambda$ but $(xy)_{\min\{s,t\}} \overline{\in \vee q_\delta^k} \lambda$. Then $\lambda(x) \geq s$, $\lambda(xy) < \min\{s,t\}$ and $\lambda(xy)+\min\{s,t\} \leq \delta-k$. This implies that $\lambda(xy) < \lambda(x)$ and $\lambda(xy) < \frac{\delta-k}{2}$. Thus $\lambda(xy) < \min\{\lambda(x), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $x_s \in \lambda$ and $y \in N$ implies $(xy)_t \in \vee q_\delta^k \lambda$.

($c4$) $\Rightarrow$ ($c4'$): Let $x, y, z \in N$ and $t \in (0,1]$ such that $z_t \in \lambda$. Then $(x(y+z)-xy)_t \in \vee q_\delta^k \lambda$. Suppose that $\lambda(x(y+z)-xy) < \min\{\lambda(z), \frac{\delta-k}{2}\}$. Choose t such that $\lambda(x(y+z)-xy) < t \leq \min\{\lambda(z), \frac{\delta-k}{2}\}$. Then $\lambda(z) \geq t$ and $\lambda(x(y+z)-xy) < t$. Since $t \leq \frac{\delta-k}{2}$, we have $\lambda(x(y+z)-xy)+t < 2t \leq \delta-k$. That is, $(x(y+z)-xy)_t \overline{\in} \lambda$ and $(x(y+z)-xy)_t \overline{q_\delta^k} \lambda$. This implies that $(x(y+z)-xy)_t \overline{\in \vee q_\delta^k} \lambda$, which is a contradiction. Hence $\lambda(x(y+z)-xy) \geq \min\{\lambda(z), \frac{\delta-k}{2}\}$.

($c4'$) $\Rightarrow$ ($c4$): Let $x, y, z \in N$ and $t \in (0,1]$ be such that $z_t \in \lambda$ but $(x(y+z)-xy)_t \overline{\in \vee q_\delta^k} \lambda$. Then $\lambda(z) \geq t$, $\lambda(x(y+z)-xy) < t$ and $\lambda(x(y+z)-xy)+t \leq \delta-k$. This implies that $\lambda(x(y+z)-xy) < \lambda(z)$ and $\lambda(x(y+z)-xy) < \frac{\delta-k}{2}$. Thus $\lambda(x(y+z)-xy) < \min\{\lambda(z), \frac{\delta-k}{2}\}$, which is a contradiction. Therefore $z_t \in \lambda$ and $x, y \in N$ implies $(x(y+z)-xy)_t \in \vee q_\delta^k \lambda$. □

Theorem 3.18. *A fuzzy subset* λ *of* N *is an* $(\in, \in \vee q_\delta^k)$*-fuzzy ideal of* N *if and only if for all* $x, y, z \in N$,

(i) $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$,
(ii) $\lambda(y+x-y) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$,
(iii) $\lambda(xy) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$,
(iv) $\lambda(x(y+z)-xy) \geq \min\{\lambda(z), \frac{\delta-k}{2}\}$.

Proof. The proof follows from Lemma 3.17. □

Remark 3.19. *It is obvious that every* $(\in, \in \vee q_\delta^k)$*-fuzzy ideal of* N *is an* $(\in, \in \vee q_\delta^k)$*-fuzzy subnear-ring of* N.

If we take $\delta = 1$ and $k = 0$ in Theorem 3.18, we have the following corollary.

Corollary 3.20. ([21], *Theorem 3.8*) *A fuzzy subset* λ *of* N *is an* $(\in, \in \vee q)$*-fuzzy ideal of* N *if and only if for all* $x, y, z \in N$,

(i) $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$,

(ii) $\lambda(y+x-y) \geq \min\{\lambda(x), 0.5\}$,
(iii) $\lambda(xy) \geq \min\{\lambda(x), 0.5\}$,
(iv) $\lambda(x(y+z)-xy) \geq \min\{\lambda(z), 0.5\}$.

If we take $\delta = 1$ in Theorem 3.18, we have the following corollary.

Corollary 3.21. ([6], *Lemma 3.11*) *A fuzzy subset λ of N is an $(\in, \in \vee q_k)$-fuzzy ideal of N if and only if for all $x, y, z \in N$,*

(i) $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$,
(ii) $\lambda(y+x-y) \geq \min\{\lambda(x), \frac{1-k}{2}\}$,
(iii) $\lambda(xy) \geq \min\{\lambda(x), \frac{1-k}{2}\}$,
(iv) $\lambda(x(y+z)-xy) \geq \min\{\lambda(z), \frac{1-k}{2}\}$.

If we take $k = 0$ in Theorem 3.18, we have the following corollary.

Corollary 3.22. *A fuzzy subset λ of N is an $(\in, \in \vee q_\delta)$-fuzzy ideal of N if and only if for all $x, y, z \in N$,*

(i) $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$,
(ii) $\lambda(y+x-y) \geq \min\{\lambda(x), \frac{\delta}{2}\}$,
(iii) $\lambda(xy) \geq \min\{\lambda(x), \frac{\delta}{2}\}$,
(iv) $\lambda(x(y+z)-xy) \geq \min\{\lambda(z), \frac{\delta}{2}\}$.

Remark 3.23. *For any $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) λ of N, we can conclude that*

(i) if $\lambda(x) < \frac{\delta-k}{2}$ for all $x \in N$, then λ is a fuzzy subnear-ring (resp. ideal) of N,
(ii) if $\delta = 1$ and $k = 0$, then λ is an $(\in, \in \vee q)$-fuzzy subnear-ring (resp. ideal) of N,
(iii) if $\delta = 1$, then λ is an $(\in, \in \vee q_k)$-fuzzy subnear-ring (resp. ideal) of N,
(iv) if $k = 0$, then λ is an $(\in, \in \vee q_\delta)$-fuzzy subnear-ring (resp. ideal) of N.

Remark 3.24. *Every fuzzy ideal, $(\in, \in \vee q)$-fuzzy ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_\delta)$-fuzzy ideal of N is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N, but the converses are not necessarily true as shown in the following example.*

Example 3.25. *Consider the near-ring N as defined in Example 3.15. Define a fuzzy subset λ of N by $\lambda(0) = 0.45$, $\lambda(a) = 0.8$, $\lambda(b) = 0.4$ and $\lambda(c) = 0.65$. Then λ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N with $\delta = 0.9$, and $k = 0.1$. But*

(i) *λ is not a fuzzy ideal of N, since* $0.45 = \lambda(0) = \lambda(a-a) \ngeq \min\{\lambda(a), \lambda(a)\} = \lambda(a) = 0.8$.
(ii) *λ is not an $(\in, \in \vee q)$-fuzzy ideal of N, since $a_{0.5} \in \lambda$ and $(a-a)_{0.5} \overline{\in \vee q} \lambda$.*
(iii) *λ is not an $(\in, \in \vee q_k)$-fuzzy ideal of N, since $a_{0.7} \in \lambda$, $c_{0.5} \in \lambda$ and $(a-c)_{0.5} \overline{\in \vee q_k} \lambda$.*
(iv) *λ is not an $(\in, \in \vee q_\delta)$-fuzzy ideal of N, since $a_{0.7} \in \lambda$, $c_{0.5} \in \lambda$ and $(a-c)_{0.5} \overline{\in \vee q_\delta} \lambda$.*

Theorem 3.26. *Let λ be an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N such that $\lambda(x) < \frac{\delta-k}{2}$ for all $x \in N$. Then λ is a fuzzy subnear-ring (resp. ideal) of N.*

Proof. Let $x, y, z \in N$. Then by Theorem 3.18 and the hypothesis, we have (i) $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\} = \min\{\lambda(x), \lambda(y)\}$, (ii) $\lambda(y+x-y) \geq \min\{\lambda(x), \frac{\delta-k}{2}\} = \lambda(x)$, (iii) $\lambda(xy) \geq \min\{\lambda(x), \frac{\delta-k}{2}\} = \lambda(x)$ and (iv) $\lambda(x(y+z)-xy) \geq \min\{\lambda(z), \frac{\delta-k}{2}\} = \lambda(z)$. Hence λ is a fuzzy ideal of N. □

Theorem 3.27. *Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-rings (resp. ideals) of N. Then $\lambda = \bigcap_{i\in I} \lambda_i$, is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N, where $(\bigcap_{i\in I} \lambda_i)(x) = \inf_{i\in I}\{\lambda_i(x)\}$.*

Proof. Let $x, y \in N$. Then

$$\lambda(x-y) = (\bigcap_{i\in I}\lambda_i)(x-y) = \inf_{i\in I}\{\lambda_i(x-y)\} \geq \inf_{i\in I}\{\min\{\{\lambda_i(x), \lambda_i(y), \tfrac{\delta-k}{2}\}\}\}$$
$$= \min\{\inf_{i\in I}\{\lambda_i(x)\}, \inf_{i\in I}\{\lambda_i(y)\}, \tfrac{\delta-k}{2}\} = \min\{(\bigcap_{i\in I}\lambda_i)(x), (\bigcap_{i\in I}\lambda_i)(y), \tfrac{\delta-k}{2}\} = \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\}$$

and

$$\lambda(xy) = (\bigcap_{i\in I}\lambda_i)(xy) = \inf_{i\in I}\{\lambda_i(xy)\} \geq \inf_{i\in I}\{\min\{\lambda_i(x), \lambda_i(y), \tfrac{\delta-k}{2}\}\}\}$$
$$= \min\{\inf_{i\in I}\{\lambda_i(x)\}, \inf_{i\in I}\{\lambda_i(y)\}, \tfrac{\delta-k}{2}\} = \min\{(\bigcap_{i\in I}\lambda_i)(x), (\bigcap_{i\in I}\lambda_i)(y), \tfrac{\delta-k}{2}\} = \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\}.$$

Thus $\lambda = \bigcap_{i\in I}\lambda_i$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N. In the same way, we can prove that λ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N. □

Remark 3.28. *Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideals) of N. Then $\bigcup_{i\in I}\lambda_i$ need not be an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N as shown in the following example.*

Example 3.29. *Consider the near-ring N as defined in Example 3.15. Define the fuzzy subsets λ and θ of N by $\lambda(0) = 0.7$, $\lambda(a) = 0.5$, $\lambda(b) = 0.1 = \lambda(c)$ and $\theta(0) = 0.4$, $\theta(a) = 0.1 = \theta(c)$, $\theta(b) = 0.2$. Then λ and θ are $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N with $\delta = 0.8$ and $k = 0.1$. It is easy to verify that $(\lambda\cup\theta)(0) = 0.7$, $(\lambda\cup\theta)(a) = 0.5$, $(\lambda \cup \theta)(b) = (0.2)$, $(\lambda \cup \theta)(c) = 0.1$. Since $0.1 = (\lambda \cup \theta)(c) = (\lambda \cup \theta)(a - b) \ngeq \min\{(\lambda \cup \theta)(a), (\lambda \cup \theta)(b), \frac{\delta-k}{2}\} = \min\{0.5, 0.2, 0.35\} = 0.2$, $\lambda \cup \theta$ is not an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N.*

To prove $\bigcup_{i\in I}\lambda_i$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N, we need an additional condition and so we have the following theorem.

Theorem 3.30. *Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-rings (resp. ideals) of N such that $\lambda_i \subseteq \lambda_j$ or $\lambda_j \subseteq \lambda_i$ for all $i, j \in I$. Then $\lambda = \bigcup_{i\in I}\lambda_i$, is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N, where $(\bigcup_{i\in I}\lambda_i)(x) = \sup_{i\in I}\{\lambda_i(x)\}$.*

Proof. Let $x, y \in N$. Then

$$\lambda(x-y) = \bigcup_{i\in I}\lambda_i(x-y) = \sup_{i\in I}\{\lambda_i(x-y)\} \geq \sup_{i\in I}\left\{\min\left\{\lambda_i(x), \lambda_i(y), \tfrac{\delta-k}{2}\right\}\right\}$$
$$= \min\{\sup_{i\in I}\{\lambda_i(x)\}, \sup_{i\in I}\{\lambda_i(y)\}, \tfrac{\delta-k}{2}\} \qquad (D_1)$$
$$= \min\{(\bigcup_{i\in I}\lambda_i)(x), (\bigcup_{i\in I}\lambda_i)(y), \tfrac{\delta-k}{2}\} = \min\left\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\right\}$$

and

$$\lambda(xy) = \bigcup_{i\in I}\lambda_i(xy) = \sup_{i\in I}\{\lambda_i(xy)\} \geq \sup_{i\in I}\{\min\{\lambda_i(x), \lambda_i(y), \tfrac{\delta-k}{2}\}\}$$
$$= \min\{\sup_{i\in I}\{\lambda_i(x)\}, \sup_{i\in I}\{\lambda_i(y)\}, \tfrac{\delta-k}{2}\}$$
$$= \min\{(\bigcup_{i\in I}\lambda_i)(x), (\bigcup_{i\in I}\lambda_i)(y), \tfrac{\delta-k}{2}\} = \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\}.$$

Thus $\lambda = \bigcup_{i\in I}\lambda_i$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N.

In the following we prove that the equation (D_1) holds. It is obvious that $\sup_{i\in I}\left\{\min\{\lambda_i(x), \lambda_i(y), \frac{\delta-k}{2}\}\right\}$ $\leq \min\left\{\sup_{i\in I}\{\lambda_i(x)\}, \sup_{i\in I}\{\lambda_i(y)\}, \frac{\delta-k}{2}\right\}$. Suppose that $\sup_{i\in I}\left\{\min\{\lambda_i(x), \lambda_i(y), \frac{\delta-k}{2}\}\right\} < \min\left\{\sup_{i\in I}\{\lambda_i(x)\}, \sup_{i\in I}\{\lambda_i(y)\}, \frac{\delta-k}{2}\right\}$.

Then there exists t such that $\sup_{i\in I}\left\{\min\{\lambda_i(x), \lambda_i(y), \frac{\delta-k}{2}\}\right\} < t < \min\left\{\sup_{i\in I}\{\lambda_i(x)\}, \sup_{i\in I}\{\lambda_i(y)\}, \frac{\delta-k}{2}\right\}$. Since $\lambda_i \subseteq \lambda_j$ or $\lambda_j \subseteq \lambda_i$ for all $i, j \in I$, there exists k such that $t < \min\{\lambda_k(x), \lambda_k(y), \frac{\delta-k}{2}\}$. On the other hand, $\min\{\lambda_i(x), \lambda_i(y), \frac{\delta-k}{2}\} < t$ for all $i \in I$, which is a contradiction. Hence $\sup_{i\in I}\left\{\min\{\lambda_i(x), \lambda_i(y), \frac{\delta-k}{2}\}\right\} = \min\left\{\sup_{i\in I}\{\lambda_i(x)\}, \sup_{i\in I}\{\lambda_i(y)\}, \frac{\delta-k}{2}\right\}$.
In a same manner, we can prove that λ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N. □

Theorem 3.31. *Let A be a subnear-ring (resp. ideal) of N and λ be a fuzzy subset of N such that*

$$\lambda(x) = \begin{cases} \eta & if \quad x \in A, \\ 0 & otherwise, \end{cases}$$

where $\eta \geq \frac{\delta-k}{2}$. Then λ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N.

Proof. Let $x, y \in N$ and $t, r \in (0, 1]$ be such that $x_t \in \lambda$ and $y_r \in \lambda$. Then $\lambda(x) \geq t > 0$ and $\lambda(y) \geq r > 0$. This implies that $\lambda(x) = \eta = \lambda(y)$. So $x, y \in A$. Therefore $x - y \in A$ and so $\lambda(x - y) = \eta$. If $\min\{t, r\} \leq \frac{\delta-k}{2}$, then $\lambda(x - y) = \eta \geq \frac{\delta-k}{2} \geq \min\{t, r\}$. Hence $(x - y)_{\min\{t,r\}} \in \lambda$. If $\min\{t, r\} > \frac{\delta-k}{2}$, then $\lambda(x - y) + \min\{t, r\} \geq \frac{\delta-k}{2} + \min\{t, r\} > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$ and so $(x - y)_{\min\{t,r\}} q_\delta^k \lambda$. Therefore $(x - y)_{\min\{t,r\}} \in \vee q_\delta^k \lambda$. Let $x, y \in N$ and $t \in (0, 1]$ be such that $x_t \in \lambda$. Then $\lambda(x) \geq t > 0$. This implies that $x \in A$ and so $y + x - y \in A$. Thus $\lambda(y + x - y) = \eta$. If $t \leq \frac{\delta-k}{2}$, then $\lambda(y + x - y) = \eta \geq \frac{\delta-k}{2} \geq t$. Hence $(y + x - y)_t \in \lambda$. If $t > \frac{\delta-k}{2}$, then $\lambda(y + x - y) + t \geq \frac{\delta-k}{2} + t > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$ and so $(y + x - y)_t q_\delta^k \lambda$. Therefore $(y + x - y)_t \in \vee q_\delta^k \lambda$. Let $x, y \in N$ and $t \in (0, 1]$ be such that $x_t \in \lambda$. Then $\lambda(x) \geq t > 0$ which implies $x \in A$ and so $xy \in A$, that is, $\lambda(xy) = \eta$. If $t \leq \frac{\delta-k}{2}$, then $\lambda(xy) = \eta \geq \frac{\delta-k}{2} \geq t$. Hence $(xy)_t \in \lambda$. If $t > \frac{\delta-k}{2}$, then $\lambda(xy) + t \geq \frac{\delta-k}{2} + t > \frac{\delta-k}{2} + \frac{\delta-k}{2} = \delta - k$ and so $(xy)_t q_\delta^k \lambda$. Therefore $(xy)_t \in \vee q_\delta^k \lambda$. Similarly, we can prove that $(x(y + z) - xy)_t \in \vee q_\delta^k \lambda$ for all $x, y \in N$ and $z \in A$. Hence λ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N. □

Definition 3.32. *Let A be a subset of N. Then the fuzzy subset λ_A in N defined by*

$$\lambda_A(x) = \begin{cases} \delta - k & if\, x \in A, \\ 0 & otherwise, \end{cases}$$

where $x \in N$, is called a (δ, k)-characteristic fuzzy subset of A in N.

Theorem 3.33. *A non-empty subset A of N is a subnear-ring (resp. ideal) of N if and only if the (δ, k)-characteristic fuzzy subset λ_A of A is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N.*

Proof. Let A be an ideal of N. Then by Theorem 3.31, λ_A is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N.

Conversely, assume that λ_A is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N. Let $a, b \in A$. Then $\lambda_A(a - b) \geq \min\{\lambda_A(a), \lambda_A(b), \frac{\delta-k}{2}\} = \min\{\delta - k, \delta - k, \frac{\delta-k}{2}\} = \frac{\delta-k}{2}$. This implies that $a - b \in A$. Let $a \in A$ and $b \in N$. Then $\lambda_A(b + a - b) \geq \min\{\lambda_A(a), \frac{\delta-k}{2}\} = \frac{\delta-k}{2}$. So $b + a - b \in A$. Let $a \in A$ and $b \in N$. Then $\lambda_A(ab) \geq \min\{\lambda_A(a), \frac{\delta-k}{2}\} = \frac{\delta-k}{2}$. Thus $ab \in A$. Let $a, b \in N$ and $c \in A$. Then $\lambda_A(a(b + c) - ab) \geq \min\{\lambda_A(c), \frac{\delta-k}{2}\} = \frac{\delta-k}{2}$. So $a(b + c) - ab \in A$. Hence A is an ideal of N. □

Using Definition 3.32 with $\delta = 1$ and $k = 0$ in Theorem 3.33, we have the following corollaries.

Corollary 3.34. ([21], *Theorem 3.12) A non-empty subset A of N is a subnear-ring (resp. ideal) of N if and only if the characteristic function χ_A of A is an $(\in, \in \vee q)$-fuzzy subnear-ring (resp. ideal) of N.*

Corollary 3.35. ([6], *Theorem 3.17) A non-empty subset A of N is a subnear-ring (resp. ideal) of N if and only if the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy subnear-ring (resp. ideal) of N.*

Corollary 3.36. *A non-empty subset A of N is a subnear-ring (resp. ideal) of N if and only if the characteristic function χ_A of A is an $(\in, \in \vee q_\delta)$-fuzzy subnear-ring (resp. ideal) of N.*

Theorem 3.37. *A fuzzy subset λ of N is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N if and only if the non-empty level subset λ_t of λ is an subnear-ring (resp. ideal) of N for all $t \in (0, \frac{\delta-k}{2}]$.*

Proof. Assume that λ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N. Let $t \in (0, \frac{\delta-k}{2}]$ and $x, y, z \in \lambda_t$. Then $\lambda(x) \geq t$, $\lambda(y) \geq t$ and $\lambda(z) \geq t$. It follows that $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\} \geq \min\{t, t, \frac{\delta-k}{2}\} = t$ and so $x - y \in \lambda_t$, $\lambda(y + x - y) \geq \min\{\lambda(x), \frac{\delta-k}{2}\} \geq \min\{t, \frac{\delta-k}{2}\} = t$ and so $y + x - y \in \lambda_t$, $\lambda(xy) \geq \min\{\lambda(x), \frac{\delta-k}{2}\} \geq \min\{t, \frac{\delta-k}{2}\} = t$ and so $xy \in \lambda_t$ and $\lambda(x(y+z) - xy) \geq \min\{\lambda(z), \frac{\delta-k}{2}\} \geq \min\{t, \frac{\delta-k}{2}\} = t$ and so $x(y+z) - xy \in \lambda_t$. Thus λ_t is an ideal of N.

Conversely, let λ be a fuzzy subset of N such that λ_t is an ideal of N for all $t \in (0, \frac{\delta-k}{2}]$. Let $x, y, z \in N$. Suppose $\lambda(x - y) < \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Choose t such that $\lambda(x-y) < t \leq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. This implies that $x, y \in \lambda_t$ and $x - y \notin \lambda_t$, which is a contradiction. Thus $\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in N$. Suppose $\lambda(y + x - y) < \min\{\lambda(x), \frac{\delta-k}{2}\}$. Choose t such that $\lambda(y + x - y) < t \leq \min\{\lambda(x), \frac{\delta-k}{2}\}$. Then $x \in \lambda_t$ and $y + x - y \notin \lambda_t$, which is a contradiction. Thus $\lambda(y + x - y) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$ for all $x, y \in N$. Similarly, it can be shown that $\lambda(xy) \geq \min\{\lambda(x), \frac{\delta-k}{2}\}$ for all $x, y \in N$ and $\lambda(x(y+z) - xy) \geq \min\{\lambda(z), \frac{\delta-k}{2}\}$ for all $x, y, z \in N$. Thus λ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N. □

Remark 3.38. *Let λ be an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N. Then the level subset λ_t may not be a subnear-ring (resp. ideal) of N for all $t \in (\frac{\delta-k}{2}, 1]$ as shown in the following example.*

Example 3.39. *Consider the $(\in, \in \vee q_\delta^k)$-fuzzy ideal λ of N as defined in Example 3.15. Choose $t = 0.7 \in (0.3, 1] = (\frac{\delta-k}{2}, 1]$. Then the level subset $\lambda_{0.7}$ is not an ideal of N, because $c + c = 0 \notin \lambda_{0.7} = \{c\}$.*

If we take $\delta = 1$ and $k = 0$ in Theorem 3.37, we have the following corollary.

Corollary 3.40. ([21], *Theorem 3.13) A fuzzy subset λ of N is an $(\in, \in \vee q)$-fuzzy subnear-ring (resp. ideal) of N if and only if the non-empty level subset λ_t of λ is a subnear-ring (resp. ideal) of N for all $t \in (0, 0.5]$.*

If we take $\delta = 1$ in Theorem 3.37, we have the following corollary.

Corollary 3.41. ([6], *Theorem 3.20) A fuzzy subset λ of N is an $(\in, \in \vee q_k)$-fuzzy subnear-ring (resp. ideal) of N if and only if the non-empty level subset λ_t of λ is a subnear-ring (resp. ideal) of N for all $t \in (0, \frac{1-k}{2}]$.*

If we take $k = 0$ in Theorem 3.37, we have the following corollary.

Corollary 3.42. *A fuzzy subset λ of N is an $(\in, \in \vee q_\delta)$-fuzzy subnear-ring (resp. ideal) of N if and only if the non-empty level subset λ_t of λ is a subnear-ring (resp. ideal) of N for all $t \in (0, \frac{\delta}{2}]$.*

Definition 3.43. *[20] Let λ be a fuzzy subset of a set X and $t \in (0, 1]$. Then the set $[\lambda_t]_\delta^k = \{x \in X | x_t q_\delta^k \lambda\} = \{x \in X | \lambda(x) + t > \delta - k\}$ is called a q_δ^k-level subset of λ in X.*

Theorem 3.44. *If a fuzzy subset λ of N is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N, then the non-empty level subset $[\lambda_t]_\delta^k$ of λ is a subnear-ring (resp. ideal) of N for all $t \in (\frac{\delta-k}{2}, 1]$.*

Proof. Assume that λ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N. Let $t \in (\frac{\delta-k}{2}, 1]$ and $x, y \in [\lambda_t]_\delta^k$. Then $\lambda(x) + t > \delta - k$ and $\lambda(y) + t > \delta - k$ and so $\lambda(x) > \delta - k - t$ and $\lambda(y) > \delta - k - t$. Since $\frac{\delta-k}{2} < t \leq 1$, we have $-t < -\frac{\delta-k}{2}$ and so $\delta - k - t < \frac{\delta-k}{2}$. Then $\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\} > \min\{\delta - k - t, \delta - k - t, \delta - k - t\} = \delta - k - t$. This implies that $\lambda(x - y) > \delta - k - t$ and so $\lambda(x - y) + t > \delta - k$. Thus $x - y \in [\lambda_t]_\delta^k$. Similarly, we can prove the other conditions of ideal hold. Hence $[\lambda_t]_\delta^k$ is an ideal of N for all $t \in (\frac{\delta-k}{2}, 1]$. □

Definition 3.45. *Let λ be a fuzzy subset of a set X and $t \in (0,1]$. Then the set $\Omega(\lambda;t) = \{x \in X | x_t \in \vee q^k_\delta \lambda\} = \{x \in X | \lambda(x) \geq t$ or $\lambda(x) + t > \delta - k\}$ is called an $\in \vee q^k_\delta$-level subset of λ in X.*

Obviously $\Omega(\lambda;t) = \lambda_t \cup [\lambda_t]^k_\delta$.

Theorem 3.46. *A fuzzy subset λ of N is an $(\in, \in \vee q^k_\delta)$-fuzzy subnear-ring (resp. ideal) of N if and only if the non-empty level subset $\Omega(\lambda;t)$ of λ is a subnear-ring (resp. ideal) of N for all $t \in (0,1]$.*

Proof. Let λ be an $(\in, \in \vee q^k_\delta)$-fuzzy subnear-ring of N and let $x, y \in \Omega(\lambda;t)$ for all $t \in (0,1]$. Then $x_t \in \vee q^k_\delta \lambda$ and $y_t \in \vee q^k_\delta \lambda$, that is, $\lambda(x) \geq t$ or $\lambda(x) + t > \delta - k$ and $\lambda(y) \geq t$ or $\lambda(y) + t > \delta - k$. Thus $\lambda(x) \geq t$ or $\lambda(x) > \delta - k - t \geq \delta - k - 1$ and $\lambda(y) \geq t$ or $\lambda(y) > \delta - k - t \geq \delta - k - 1$. If $t \in (0, \frac{\delta-k}{2}]$, then $0 < t \leq \frac{\delta-k}{2}$. This implies that $\delta - k - t \geq \frac{\delta-k}{2} \geq t$. Hence it follows from the above that $\lambda(x) \geq t$ and $\lambda(y) \geq t$. By hypothesis $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\} \geq \min\{t,t,t\} = t$ and so $(x-y)_t \in \lambda$. Thus $x - y \in \Omega(\lambda;t)$. If $t \in (\frac{\delta-k}{2}, 1]$, then $\frac{\delta-k}{2} < t \leq 1$. Since $-t < -\frac{\delta-k}{2}$, we have $\delta - k - t < \frac{\delta-k}{2} < t$. Hence $\lambda(x) > \delta - k - t$ and $\lambda(y) > \delta - k - t$. By hypothesis $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\} \geq \min\{\delta-k-t, \delta-k-t, \delta-k-t\} = \delta-k-t$ and so $\lambda(x-y)+t > \delta-k$. This implies that $x - y \in \Omega(\lambda;t)$. Similarly, we can prove the remaining conditions of ideal hold. Hence $\Omega(\lambda;t)$ is an ideal of N for all $t \in (0,1]$. Conversely, assume that λ is a fuzzy subset of N and $t \in (0,1]$ such that $\Omega(\lambda;t)$ is an ideal of N. Suppose $x, y \in N$ such that $\lambda(x-y) < \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Choose $t \in (0,1]$ such that $\lambda(x-y) < t \leq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$. Then $x \in \Omega(\lambda;t)$, $y \in \Omega(\lambda;t)$ and $(x-y) \notin \Omega(\lambda;t)$. This contradicts our hypothesis. Therefore $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta-k}{2}\}$ for all $x, y \in N$. Similary, we can prove the remaining conditions of $(\in, \in \vee q^k_\delta)$-fuzzy ideal hold. Hence λ is an $(\in, \in \vee q^k_\delta)$-fuzzy ideal on N. □

4. Homomorphic image and pre-image of $(\in, \in \vee q^k_\delta)$-fuzzy ideals

In this section, we discuss the homomorphic image and pre-image of $(\in, \in \vee q^k_\delta)$-fuzzy subnear-rings and $(\in, \in \vee q^k_\delta)$-fuzzy ideals of near-rings.

Definition 4.1. *[24] Let f be a mapping from a set X into a set Y. If λ is a fuzzy subset of X, then the image of λ under f, denoted by $f(\lambda)$, is a fuzzy subset of Y defined by*

$$f(\lambda)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \lambda(x) & if \quad f^{-1}(y) \neq \emptyset, \\ 0 & otherwise, \end{cases}$$

for every $y \in Y$, where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Definition 4.2. *[24] A fuzzy subset λ of a set X is said to have sup property if, for any subset A of X, there exists $a_0 \in A$ such that $\lambda(a_0) = \sup\limits_{a \in A} \lambda(a)$.*

Theorem 4.3. *Let $f : N_1 \to N_2$ be a homomorphism between near-rings N_1 and N_2. If λ is an $(\in, \in \vee q^k_\delta)$-fuzzy subnear-ring (resp. ideal) of N_1 with sup property, then $f(\lambda)$ is an $(\in, \in \vee q^k_\delta)$-fuzzy subnear-ring (resp. ideal) of N_2.*

Proof. Let $x, y \in N_2$. Suppose either $x \notin f(N_1)$ or $y \notin f(N_1)$. Then $\min\{f(\lambda)(x), f(\lambda)(y)\} = 0 \leq f(\lambda)(x-y)$. Suppose $x \in f(N_1)$ and $y \in f(N_1)$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in N_1$. Since $f(a), f(b) \in f(N_1)$, we have $a_0 \in f^{-1}[f(a)]$ and $b_0 \in f^{-1}[f(b)]$ such that $\lambda(a_0) = \sup\limits_{a_1 \in f^{-1}[f(a)]} \lambda(a_1)$ and $\lambda(b_0) = \sup\limits_{b_1 \in f^{-1}[f(b)]} \lambda(b_1)$ respectively.

Then

$$\begin{aligned} f(\lambda)(x-y) &= f(\lambda)(f(a-b)) = \sup_{z \in f^{-1}[f(a-b)]} \lambda(z) \geq \lambda(a_0 - b_0) \geq \min\{\lambda(a_0), \lambda(b_0), \tfrac{\delta-k}{2}\} \\ &= \min\{\sup_{a_1 \in f^{-1}[f(a)]} \lambda(a_1), \sup_{b_1 \in f^{-1}[f(b)]} \lambda(b_1), \tfrac{\delta-k}{2}\} = \min\{f(\lambda)(f(a)), f(\lambda)(f(b)), \tfrac{\delta-k}{2}\} \\ &= \min\{f(\lambda)(x), f(\lambda)(y), \tfrac{\delta-k}{2}\} \end{aligned}$$

and

$$\begin{aligned} f(\lambda)(xy) = f(\lambda)(f(ab)) &= \sup_{z\in f^{-1}[f(ab)]} \lambda(z) \geq \lambda(a_0b_0) \geq \min\{\lambda(a_0), \lambda(b_0), \tfrac{\delta-k}{2}\} \\ &= \min\{\sup_{a_1\in f^{-1}[f(a)]} \lambda(a_1), \sup_{b_1\in f^{-1}[f(b)]} \lambda(b_1), \tfrac{\delta-k}{2}\} = \min\{f(\lambda)(f(a)), f(\lambda)(f(b)), \tfrac{\delta-k}{2}\} \\ &= \min\{f(\lambda)(x), f(\lambda)(y), \tfrac{\delta-k}{2}\}. \end{aligned}$$

Hence $f(\lambda)$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N_2. In a similar manner, we can prove $f(\lambda)$ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N_2. □

Definition 4.4. *[24] Let f be a mapping from a set X into a set Y. If λ is a fuzzy subset of Y, then the pre-image of λ under f, denoted by $f^{-1}(\lambda)$, is a fuzzy subset of X defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$ for all $x \in X$.*

Theorem 4.5. *Let $f : N_1 \to N_2$ be a homomorphism between near-rings N_1 and N_2. If λ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N_2, then $f^{-1}(\lambda)$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N_1.*

Proof. Let $a, b \in N_1$. Then

$$f^{-1}(\lambda)(a-b) = \lambda(f(a-b)) = \lambda(f(a) - f(b)) \geq \min\{\lambda(f(a)), \lambda(f(b)), \tfrac{\delta-k}{2}\} = \min\{f^{-1}(\lambda(a)), f^{-1}(\lambda(b)), \tfrac{\delta-k}{2}\}$$

and

$$f^{-1}(\lambda)(ab) = \lambda(f(ab)) = \lambda(f(a)f(b)) \geq \min\{\lambda(f(a)), \lambda(f(b)), \tfrac{\delta-k}{2}\} = \min\{f^{-1}(\lambda(a)), f^{-1}(\lambda(b)), \tfrac{\delta-k}{2}\}.$$

Hence $f^{-1}(\lambda)$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N_1. In the same way, we can prove $f^{-1}(\lambda)$ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N_1. □

To prove the converse of Theorem 4.5, we need to strengthen the condition on f.

Theorem 4.6. *Let $f : N_1 \to N_2$ be an onto homomorphism between near-rings N_1 and N_2, and let λ be a fuzzy subset of N_2. If $f^{-1}(\lambda)$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N_1, then λ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N_2.*

Proof. Let $x, y \in N_2$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in N_1$. Since $f^{-1}(\lambda)$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N_1, we have

$$\begin{aligned} \lambda(x-y) = \lambda(f(a) - f(b)) = \lambda(f(a-b)) &= f^{-1}(\lambda)(a-b) \geq \min\{f^{-1}(\lambda)(a), f^{-1}(\lambda)(b), \tfrac{\delta-k}{2}\} \\ &= \min\{\lambda(f(a)), \lambda(f(b)), \tfrac{\delta-k}{2}\} = \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\} \end{aligned}$$

and

$$\begin{aligned} \lambda(xy) = \lambda(f(a)f(b)) = \lambda(f(ab)) &= f^{-1}(\lambda)(ab) \geq \min\{f^{-1}(\lambda)(a), f^{-1}(\lambda)(b), \tfrac{\delta-k}{2}\} \\ &= \min\{\lambda(f(a)), \lambda(f(b)), \tfrac{\delta-k}{2}\} = \min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\}. \end{aligned}$$

Hence λ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring of N_2. Similarly we can prove λ is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N_2. □

5. (δ, k)-upper and lower parts of $(\in, \in \vee q_\delta^k)$-fuzzy ideals

In this section, we define the notions of (δ, k)-upper and lower parts of the (δ, k)-characteristic fuzzy subset of a set and obtain some related results.

Definition 5.1. *[20] Let λ be a fuzzy subset of a set X. Then we define the (δ, k)-upper part λ^+ and the (δ, k)-lower part λ^- of λ as follows:*

$$\lambda^+ = \max\{\lambda(x), \tfrac{\delta-k}{2}\} \quad and \quad \lambda^- = \min\{\lambda(x), \tfrac{\delta-k}{2}\} \quad for\ all\ x \in X.$$

In Example 3.25, we have shown that an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N is not a fuzzy ideal of N. In the following theorem, we show that the lower part λ^- of an $(\in, \in \vee q_\delta^k)$-fuzzy ideal λ is a fuzzy ideal of N.

Theorem 5.2. *If λ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N, then the lower part λ^- of λ is a fuzzy subnear-ring (resp. ideal) of N.*

Proof. Let λ be an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N and $x, y, z \in N$. Then

$$\begin{aligned}\lambda^-(x - y) &= \min\{\lambda(x - y), \tfrac{\delta-k}{2}\} \geq \min\{\min\{\lambda(x), \lambda(y), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\}\\ &= \min\{\min\{\lambda(x), \tfrac{\delta-k}{2}\}, \min\{\lambda(y), \tfrac{\delta-k}{2}\}\} = \min\{\lambda^-(x), \lambda^-(y)\},\end{aligned}$$

$$\lambda^-(y + x - y) = \min\{\lambda(y + x - y), \tfrac{\delta-k}{2}\} \geq \min\{\min\{\lambda(x), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\} = \min\{\lambda(x), \tfrac{\delta-k}{2}\} = \lambda^-(x),$$

$$\lambda^-(xy) = \min\{\lambda(xy), \tfrac{\delta-k}{2}\} \geq \min\{\min\{\lambda(x), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\} = \min\{\lambda(x), \tfrac{\delta-k}{2}\} = \lambda^-(x)$$

and

$$\lambda^-(x(y + z) - xy) = \min\{\lambda(x(y + z) - xy), \tfrac{\delta-k}{2}\} \geq \min\{\min\{\lambda(z), \tfrac{\delta-k}{2}\}, \tfrac{\delta-k}{2}\} = \min\{\lambda(z), \tfrac{\delta-k}{2}\} = \lambda^-(z).$$

Therefore λ^- is a fuzzy ideal of N. □

Combining Theorem 5.2, Remark 3.14 and Remark 3.24, we have the follwing theorem.

Theorem 5.3. *If λ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N, then the lower part λ^- of λ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N.*

Combining Theorem 5.3 and Theorem 3.44, we have the following corollary.

Corollary 5.4. *Let λ be an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N. Then the non-empty level subset $[\lambda_t^-]_\delta^k = \{x \in N | \lambda^-(x) + t > \delta - k\}$ of λ is a subnear-ring (resp. ideal) of N for all $t \in (\frac{\delta-k}{2}, 1]$.*

Theorem 5.5. *Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-rings (resp. ideals) of N. Then $\lambda^- = \bigcap_{i\in I} \lambda_i^-$ is an $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of N where $(\bigcap_{i\in I} \lambda_i^-)(x) = \inf_{i\in I}\{\lambda_i^-(x)\}$.*

Proof. The proof is straightforward and so we omit it. □

Definition 5.6. *Let A be a non-empty subset of N. Then the (δ, k)-upper part λ_A^+ and the (δ, k)-lower part λ_A^- of the (δ, k)-characteristic fuzzy subset λ_A of A are defined by*

$$\lambda_A^+(x) = \begin{cases} \delta - k & if \quad x \in A, \\ \frac{\delta-k}{2} & if \quad x \notin A, \end{cases} \quad and \quad \lambda_A^-(x) = \begin{cases} \frac{\delta-k}{2} & if \quad x \in A, \\ 0 & if \quad x \notin A. \end{cases}$$

In the above definition, if we take $\lambda_A^+(x) = 1$ when $x \in A$, we obtain the (δ, k)-*upper part* χ_A^+ and the (δ, k)-*lower part* χ_A^- of the *characteristic function* χ_A of A.

Lemma 5.7. *Let A and B be any two non-empty subsets of N. Then the following statements are hold: $(\lambda_A \cap \lambda_B)^- = \lambda_{A\cap B}^-$, $(\lambda_A \cup \lambda_B)^- = \lambda_{A\cup B}^-$ and $(\lambda_A \circ \lambda_B)^- = \lambda_{AB}^-$.*

Proof. The proofs are straightforward and so we omit it. □

Theorem 5.8. *Let A be a non-empty subset of N. Then the (δ, k)-lower part λ_A^- of the (δ, k)-characteristic fuzzy subset λ_A of A is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N if and only if A is an ideal of N.*

Proof. It can be easily verified by the similar way to the proof of Theroem 3.33. □

Corollary 5.9. *Let A be a non-empty subset of N. Then the (δ, k)-lower part χ_A^- of the characteristic function χ_A of A is an $(\in, \in \vee q_\delta^k)$-fuzzy ideal of N if and only if A is an ideal of N.*

6. Conclusion

In this paper, we have introduced the notions of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring and $(\in, \in \vee q_\delta^k)$-fuzzy ideal of a near-ring which are generalization of $(\in, \in \vee q_k)$-fuzzy subnear-ring and $(\in, \in \vee q_k)$-fuzzy ideal respectively. We have provided the characterizations of $(\in, \in \vee q_\delta^k)$-fuzzy subnear-ring (resp. ideal) of a near-ring and discussed some related properties. In our future research, we will apply the results of this paper to other fuzzy algebraic substructures of semirings, near-rings and rings.

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