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STUDY OF FUZZY SOFT SETS WITH SOME ORDER ON SET OF PARAMETERS

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Abstract. In this day and age soft sets have become very popular in decision making problems. It is an emerging field of research, which has attracted many researchers and practitioners. It has some limitations to handle data when order exists on set of attributes. This deficiency was overcome by "Ali et.al.[5]". They defined soft sets when there is an order on set of parameters. In fact, this new structure was a lattice and was declared as lattice ordered soft sets. As this research progresses, one faces difficulties with fuzzy soft sets having certain order on set of parameters as there does not exist such type of study. In the present research, we investigate the study of fuzzy soft sets with some order on set of parameters. It will help us to deal with decision making real world problems involving lattice ordered fuzzy soft sets.

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1. Introduction:

Molodtsov [18] initiated the study of soft sets to provide a mathematical tool to handle uncertainty. On the other hand, soft sets have ability of hybridization with other theories such as fuzzy sets see [14, 19, 24]. Many authors contributed to the theoretical development of soft sets and different operations in soft sets have been studied in [2, 13, 17]. The intensive study of algebraic structures with the help of soft set was made by several authors. In this regard [1, 4, 9, 22] are valuable contributions. Soft topological spaces were studied in [23]. In decision making problems, soft sets have been applied in different contexts, see [15, 21].

The study of lattice for soft ideals of soft semigroup is made in [22]. Recently Qin and Hong [20] have thought about lattices of soft sets based on new approaches and operations. Ali et al. in [3], focused on algebraic structures like semigroups, semi rings, lattices and monoids of soft sets pertaining to new operations. Ali et al. also highlighted few basic issues in the results about the operations exhibited on soft sets. In their paper [2], several modern approaches are mentioned about soft sets which are basic and new operations. With these operations, several results are manipulated and laws are constructed. Naz and Shabir [19], defined algebraic constructions with fuzzy soft sets. In [5], Ali.et al. investigated soft sets when there is an order on a set of parameters. These sets are also known as lattice order soft sets. They developed general ideas about these sets and described the application of these ideas. This idea was so important that many researchers applied it in different fields of research, for instance, "Vimala et al.[25,26,28]" studied lattice ordered soft groups, fuzzy groups, and l-fuzzy groups. In 2018, Vimala et al. [27] applied the idea of anti-lattice ordered fuzzy soft groups with matrix operations. Mahmood et al. [16] came up with lattice ordered intuitionistic fuzzy soft sets in 2018. The present paper in the form of thesis was referenced in [16].

In this paper, for the first time ever study of fuzzy soft sets with some order on a set of parameters is taken into consideration. This paper has been systematically arranged in the following ways: In section 2, some basic definitions are given. In section 3, lattice ordered fuzzy soft set ($\mathcal{L}\sigma f s_s$), complement, intersection, union, basic union, basic intersection, twisting of $\mathcal{L}\sigma f s_s$ are defined and some pertinent properties are studied. In the penultimate section 4, concept of $\mathcal{L}\sigma f s_s$ in algebraic structures is defined and relevant results are deduced. In the last section 5, an application of $\mathcal{L}\sigma f s_s$ has been described in decision making problem.

2. Preliminaries

This section provides the essential definitions and preliminary results, regarding soft sets, semi rings and fuzzy soft sets which are needed within the sequel. For undefined terms and notions, we refer to [2] and [19]. A relation $\leq$ on a non-empty set α is said to be partial order if it is reflexive, antisymmetric and transitive. In this case, α is known as partially ordered set. In addition, if for every $a, b \in \alpha$, such that $a \neq b$, either $a \leq b$ or $b \leq a$, then $\leq$ is called total order on α and in this case α is called a totally ordered set or chain. Let α and β be two partially ordered sets, then the dictionary order on the Cartesian product $\alpha \times \beta$ is defined as: $(a,b) \leq_{\alpha\times\beta} (c,d)$ if and only if $a \leq_\alpha c$ and if $a = c$ then $b \leq_\beta d$. In this case $\leq_{\alpha\times\beta}$ is a partial order on $\alpha \times \beta$. If α and β are totally ordered, then $\leq_{\alpha\times\beta}$ is total order on $\alpha \times \beta$.

A lattice L is a poset in which for all $a, b \in L$ a set $\{a, b\}$ possesses a supremum and an infimum denoted as " $\vee$ " and " $\wedge$ " respectively. The lattice L is called bounded if it contains elements 0, 1 in such a way that for every $t \in L$, $0 \leq t$ and $t \leq 1$. If a lattice possesses any distributive law is known as distributive lattice. A structure that has *De* Morgan's lattice and if De Morgan's laws with an involution hold in it is known as bounded distributive lattice. If $(L,\wedge,\vee,^c,0,1)$ is De Morgan's algebra satisfying $j \wedge j^c \leq k \vee k^c$ for every $j, k \in L$, then $(L,\wedge,\vee,^c,0,1)$ is called Kleene algebra.

A fuzzy subset f of a non empty set χ is a function $f: \chi \rightarrow [0,1]$. Set of all fuzzy subsets of χ is denoted by $\lambda P(\chi)$. For $f, g \in \lambda P(\chi)$, $f \subseteq g$ if and only if $f(j) \leq g(j)$ for all $j \in \chi$. Further union and intersection of f and g are defined as:

$$(f \cup g)(x) = f(x) \vee g(x)$$

$$= max\{f(x), g(x)\} \text{ for all } x \in \chi$$
$$(f \cap g)(x) = f(x) \wedge g(x)$$

$$= min\{\{f(x), g(x)\} \text{ for all } x \in \chi.$$

If $f \in \lambda P(\chi)$ then complement of f is denoted by f^c and is defined as $f^c(x) = 1 - f(x)$.
An ordered semigroup (or po-semigroup) is defined to be an ordered set P and at the same time a semigroup such that for all $p_1, p_2, t \in P$, $p_1 \leq p_2 \Rightarrow p_1 t \leq p_2 t$ and $tp_1 \leq tp_2$. . In remaining paper, we will consider this $\preccurlyeq$ order on $\alpha \times \beta$.

DEFINITION 1. [17] Let χ be an initial universe, Σ be the set of all possible parameters under consideration with respect to χ and α be a subset of Σ. Then a pair (λ, α) is called a soft set over χ , where λ is a mapping $\lambda : \alpha \rightarrow P(\chi)$.

DEFINITION 2. [14] A pair (λ, α) is called a fuzzy soft set over χ , where λ is a mapping given by $\lambda : \alpha \rightarrow fP(\chi)$.

DEFINITION 3. [14] For two fuzzy soft sets (λ, α) and (ξ, β) over a common universe χ, we say that (λ, α) is a fuzzy soft subset of (ξ, β) if

1. $\alpha \subseteq \beta$ and
2. $\lambda(e) \subseteq \xi(e)$ for all $e \in \alpha$.

We write $(\lambda, \alpha) \subseteq^{\simeq} (\xi, \beta)$. In this case (ξ, β) is super set of (λ, α) .

DEFINITION 4. [6] Let (λ , α) and (ξ , β) be two fuzzy soft sets and χ be their common universe. Then (λ , α) is called a fuzzy soft twisted subset of (ξ , β) if

$\alpha \subseteq \beta$ and
$\xi(u) \subseteq \lambda(u)$ for all $u \in \alpha$.
It is represented by $(\lambda, \alpha) \tilde{\subseteq} (\xi, \beta)$.

DEFINITION 5. [14] Two fuzzy soft sets (λ, α) and (ξ, β) over a common universe χ are said to be fuzzy soft equal if (λ, α) is a fuzzy soft subset of (ξ, β) and (ξ, β) is a fuzzy soft subset of (λ, α).

DEFINITION 6. [30] Let χ be an initial universe set, Σ be the set of parameters and $\alpha \subseteq \Sigma$, then:

1. (λ, α) is called a relative null fuzzy soft set (with respect to α), denoted by φ_α , if $\lambda(e) = \varphi$ for all $e \in \alpha$ where φ is the subset of χ mapping every element of χ on 0.
2. (ξ, α) is called a relative whole fuzzy soft set (with respect to α), denoted by α , if $\xi(e) = 1$ for all $e \in \alpha$, where 1 is the fuzzy subset of χ which maps every element of χ on 1.

The relative whole soft set with respect to the set of parameters Σ is called the absolute soft set over χ and is simply denoted by Σ.

The relative whole fuzzy soft set with respect to the set of parameters Σ is called the absolute fuzzy soft set over ω and is represented by Σ.

DEFINITION 7. [14] Let (λ, α) and (ξ, β) be any two fuzzy soft sets over a common universe χ . Then the basic union of (λ, α) and (ξ, β) is defined as the fuzzy soft set $(\psi, \gamma) = (\lambda, \alpha) \vee (\xi, \beta)$, where $\gamma = \alpha \times \beta$ and $\psi(a, b) = \lambda(a) \cup \xi(b)$ for all $(a, b) \in \alpha \times \beta$.

DEFINITION 8. [14] Let (λ, α) and (ξ, β) be any fuzzy soft sets over a common universe χ . Then the basic intersection of these sets is defined as the fuzzy soft set $(\psi, \gamma) = (\lambda, \alpha) \wedge (\xi, \beta)$, where $\gamma = \alpha \times \beta$, and $\psi(x, t) = \lambda(x) \cap \xi(t)$ for all $(x, t) \in \alpha \times \beta$.

DEFINITION 9. [2] The extended union of two fuzzy soft sets (λ, α) and (ξ, β) over the common universe χ is the fuzzy soft set (ψ, γ), where $\gamma = \alpha \cup \beta$ and for all $k \in \gamma$,

$$\psi(k) = \begin{cases} \lambda(k) & if\ k \in \alpha - \beta \\ \xi(k) & if\ k \in \beta - \alpha \\ \lambda(k) \cup \xi(k) & if\ k \in \alpha \cap \beta. \end{cases}$$

It is denoted by $(\lambda, \alpha) \cup_\Sigma (\xi, \beta) = (\psi, \gamma)$.

DEFINITION 10. [2] Let χ be a common universe for two fuzzy soft sets (λ, Y) and (ξ, Z). Then the extended intersection of these sets is the fuzzy soft set (ψ, γ) where $\gamma = Y \cup Z$ and for all $v \in \gamma$,

$$\psi(v) = \begin{cases} \lambda(v) & if\ v \in Y - Z \\ \xi(v) & if\ v \in Z - Y \\ \lambda(v) \cap \xi(v) & if\ v \in Y \cap Z. \end{cases}$$

It is written as $(\lambda, Y) \cap_\Sigma (\xi, Z) = (\psi, \gamma)$.

DEFINITION 11. [30] Let (λ, Y) and (ξ, Z) be two fuzzy soft sets over the same universe χ, such that $Y \cap Z \neq \varphi$. Then the restricted union of (λ, Y) and (ξ, Z) is denoted by $(\lambda, Y) \cup_R (\xi, Z)$ and is defined as $(\lambda, Y) \cup_R (\xi, Z) = (\psi, \gamma)$, where $\gamma = Y \cap Z$ and for all $v \in \gamma$, $\psi(v) = \lambda(v) \cup \xi(v)$. If $Y \cap Z = \varphi$, then $(\lambda, Y) \cup_R (\xi, Z) = \varphi_\varphi$.

DEFINITION 12. [2] Let (λ, α) and (ξ, β) be two fuzzy soft sets over the same universe χ such that $\alpha \cap \beta \neq \varphi$. Then the restricted intersection of (λ, α) and (ξ, β) is denoted by $(\lambda, \alpha) \cap_R (\xi, \beta)$ and is defined as $(\lambda, \alpha) \cap_R (\xi, \beta) = (\psi, \gamma)$ where $\gamma = \alpha \cap \beta$ and for all $e \in \gamma$, $\psi(e) = \lambda(e) \cap \xi(e)$.
$(\lambda, \alpha) \cap_R (\xi, \beta) = \varphi_\varphi$, if $\alpha \cap \beta = \varphi$.

DEFINITION 13. [16] Let Σ be a set of parameters and $V, W \subseteq \Sigma$. For $(v, u) \in V \times W$, $(v \text{ and } u)$ is called the conjunction and $(v \text{ or } u)$ is known as disjunction parameter of an ordered pair (v, u) and represented by $(v \wedge^- u)$ and $(v \vee^- u)$ respectively.
We represent

$$V \oplus W = \{(v \vee^- u) \ : \ (v, u) \in V \times W\}.$$

Also

$$V \otimes W = \{(v \wedge^- u) \ : \ (v, u) \in V \times W\}.$$

DEFINITION 14. [6] Let (λ, α) and (ξ, β) be two fuzzy soft sets over the same universe χ. Then $(\lambda, \alpha) \cup_\vee (\xi, \beta) = (\psi, \gamma)$, where $\gamma = \alpha \oplus \beta$ and for all $a \vee^- b \in \gamma$, $\psi(a \vee^- b) = \lambda(a) \cup \xi(b)$.

DEFINITION 15. [6] Let (λ, α) and (ξ, β) be two fuzzy soft sets over the same universe χ . Then $(\lambda, \alpha) \cap_\wedge (\xi, \beta) = (\psi, \gamma)$, where $\gamma = \alpha \otimes \beta$ and for all $a \wedge^- b \in \gamma$, $\psi(a \wedge^- b) = \lambda(a) \cap \xi(b)$.

DEFINITION 16. [23] The complement of a fuzzy soft set (λ, α) is dubbed by $(\lambda, \alpha)^c$ and is defined by $(\lambda, \alpha)^c = (\lambda^c, \alpha)$, where $\lambda^c : \alpha \to \lambda P(\chi)$ is a mapping given by $\lambda^c(a) = 1 - \lambda(a)$, for all $a \in \alpha$.
Let us call λ^c to be the fuzzy soft complement function of λ. Clearly, $(\lambda^c)^c$ is same as λ and $((\lambda, \alpha)^c)^c = (\lambda, \alpha)$.

PROPOSITION 1.[19] Let (λ, α) and (ξ, β) be two fuzzy soft sets over the same universe χ. Then

1. $((\lambda, \alpha)^c)^c = (\lambda, \alpha)$.
2. $(\lambda, \alpha) \hat{\subset} (\xi, \alpha)$ implies $(\xi, \alpha)^c \hat{\subset} (\lambda, \alpha)^c$

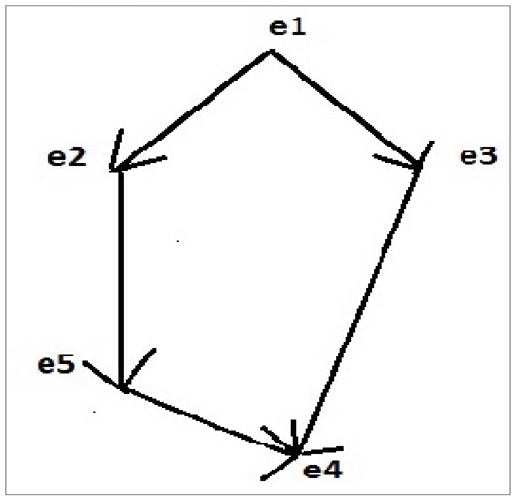
3. Lattice (anti-lattice) ordered fuzzy soft sets

It is usual that $fP(\chi)$ is a bounded lattice under union and intersection, or equivalently having set inclusion with partial order.

DEFINITION 17. A fuzzy soft set which is (λ,α) is called a lattice (anti-lattice) ordered fuzzy soft set, if a mapping $\lambda:\alpha\to fP(\chi),\ x\leq y$ implies $\lambda(x)\subseteq\lambda(y)\big(\lambda(y)\subseteq\lambda(x)\big),\ \forall\quad x,y\in\alpha$.

EXAMPLE 1. Mr. X wants to enroll his son in a private school. He visited few schools for this purpose. Let $P=\{p_1,p_2,p_3,p_4,p_5,p_6\}$ be the set of schools and $S=\{e_1$(cheaper), e_2(good environment), e_3(English medium), e_4(highly qualified staff), e_5(suggested by some friend), e_6(nearer to house)$\}$ be the set of parameters on the basis of which school will be chosen and $T=\{e_1,e_2,e_3,e_4,e_5\}\subseteq S$. The order among the elements of T depicting priority is as shown in the figure 1.

Order in elements of lattice T



Then the fuzzy soft set (λ,T) *on* P *is a fuzzy soft set as shown below:*

$$\lambda(e_1)=\left\{\frac{p_1}{0.1},\frac{p_2}{0.3},\frac{p_3}{0.4},\frac{p_4}{0.5},\frac{p_5}{0.6},\frac{p_6}{0.7}\right\}$$
$$\lambda(e_2)=\left\{\frac{p_1}{0.1},\frac{p_2}{0.4},\frac{p_3}{0.5},\frac{p_4}{0.5},\frac{p_5}{0.6},\frac{p_6}{0.7}\right\}$$
$$\lambda(e_3)=\left\{\frac{p_1}{0.2},\frac{p_2}{0.4},\frac{p_3}{0.6},\frac{p_4}{0.7},\frac{p_5}{0.6},\frac{p_6}{0.8}\right\}$$
$$\lambda(e_4)=\left\{\frac{p_1}{0.5},\frac{p_2}{0.6},\frac{p_3}{0.7},\frac{p_4}{0.8},\frac{p_5}{0.7},\frac{p_6}{0.9}\right\}$$
$$\lambda(e_5)=\left\{\frac{p_1}{0.3},\frac{p_2}{0.5},\frac{p_3}{0.7},\frac{p_4}{0.8},\frac{p_5}{0.6},\frac{p_6}{0.9}\right\}.$$

It is also shown in Table 1.

Tabular representation of (λ,T)

(λ,T)	e_1	e_2	e_3	e_4	e_5
P_1	0.1	0.1	0.2	0.5	0.3
P_2	0.3	0.4	0.4	0.6	0.5

P_3	0.4	0.5	0.6	0.7	0.7
P_4	0.5	0.5	0.7	0.8	0.8
P_5	0.6	0.6	0.6	0.7	0.6
P6	0.7	0.7	0.8	0.9	0.9

Table 1

So, for $e_1 \le e_3 \le e_4$, and $e_1 \le e_2 \le e_5 \le e_4$, we have $\lambda(e_1) \subseteq \lambda(e_3) \subseteq \lambda(e_4)$ and $\lambda(e_1) \subseteq \lambda(e_2) \subseteq \lambda(e_5) \subseteq \lambda(e_4)$ respectively. Thus the fuzzy soft set (λ, T) on P is lattice ordered.

PROPOSITION 2. On a universe χ, being (λ,α) and (ξ,β) lattice ordered fuzzy soft sets, $(\lambda,\alpha) \cap_R (\xi,\beta)$ is the largest lattice ordered fuzzy soft set over χ which is contained in both (λ,α) and (ξ,β).

Proof. Let $(\lambda,\alpha) \cap_R (\xi,\beta) = (\psi,\gamma)$, where $\gamma = \alpha \cap \beta \neq \Phi$. Then for any $f_1, f_2 \in \gamma$, $\lambda(f_1) \subseteq \lambda(f_2)$ and $\xi(f_1) \subseteq \xi\ (f_2)$ whenever $f_1 \le_\alpha f_2$ and $f_1 \le_\beta f_2$.
Also $\lambda(f_1) \cap \xi\ \ (f_1) \subseteq \lambda(f_2) \cap \xi\ \ (f_2) \Rightarrow\ \psi(f_1) \subseteq \psi(f_2)$ for $f_1 \le_\gamma f_2$.
Thus $(\lambda,\alpha) \cap_R (\xi,\beta)$ is lattice ordered fuzzy soft set.

As $(\lambda,\alpha) \cap_R (\xi,\beta) = (\psi,\gamma)$, where $\gamma = \alpha \cap \beta \neq \Phi$ and for all $f \in \gamma$, we have $\psi(f) = \lambda(f) \cap \xi(f)$.
Also, as $\lambda(f) \cap \xi(f) \subseteq \lambda(f)$ and $\lambda(f) \cap \xi(f) \subseteq \xi(f) \Rightarrow \psi(f) \subseteq \lambda(f)$ and $\psi(f) \subseteq \xi(f)$, for all $f \in \gamma$. Thus $(\psi,\gamma) \subseteq (\lambda,\alpha)$ and $(\psi,\gamma) \subseteq (\xi,\beta)$.

Let (K,D) be another lattice ordered fuzzy soft set subset of both (λ,α) and (ξ,β), that is $D \subseteq \alpha$, β and $K(i) \subseteq \lambda(i), K(i) \subseteq \xi(i)$ for all $i \in D$. Therefore $K(i) \subseteq \lambda(i) \cap \xi(i)$ for all $i \in D$, where $D \subseteq \alpha \cap \beta$. This implies $(K,D)\tilde{\subseteq}(\psi,\gamma)$.
Therefore, (ψ,γ) is the largest lattice ordered fuzzy soft set over χ which is contained in (λ,α) and (ξ,β).

PROPOSITION 3. Restricted union of (K, D) and (J,S) which are lattice (anti-lattice) ordered fuzzy soft sets is a lattice (anti-lattice) ordered fuzzy soft set.

Proof. Let (K,D) and (J,S) be lattice order fuzzy soft sets. Let $(K,D) \cup_R (J,S) = (\psi,\gamma)$, where $\gamma = D \cap S$. If $D \cap S = \varphi$, then the desired result holds trivially. Now, consider $D \cap S \neq \varphi$. Since $D, S \subseteq \Sigma$, so both D and S inherit a partial order from Σ . Therefore, for any $a_1 \le_D a_2$ we have $K(a_1) \subseteq K(a_2)$ for all $a_1, a_2 \in D$.
Also for any $b_1 \le_S b_2$ we have $J(b_1) \subseteq J(b_2)$ for all $b_1, b_2 \in D \cap S$. Therefore, for any $c_1, c_2 \in \gamma$, $K(c_1) \subseteq K(c_2)$ and $J(c_1) \subseteq J(c_2)$.

Also, $K(c_1) \cup J(c_1) \subseteq\ K(c_2) \cup J(c_2)$ which implies $\psi(c_1) \subseteq \psi(c_2)$ for $c_1 \le_\gamma c_2$. Thus $(K,D) \cup_R (J,S)$ is also a lattice order fuzzy soft set. The result also holds in the case of anti-lattice order fuzzy soft sets.

PROPOSITION 4. Let (K, D) and (J, S) be lattice (anti-lattice) ordered fuzzy soft sets and if $(K, D) \subseteq (J, S)$ or $(J, S) \subseteq (K, D)$, then their extended union is also lattice (anti-lattice) ordered fuzzy soft set.

Proof. Let us consider $(K, D) \subseteq (J, S)$ so $D \subseteq S$ and $K(e) \subseteq J(e)$ for all $e \in D$. Then extended union is given by $(K, D) \cup_\varepsilon (J, S) = (\psi, \gamma)$ where $\gamma = D \cap S$ as $D \subseteq S$ then $\gamma = S$, this implies that $\psi(c) = J(c)$ for all $c \in \gamma$. So, $(\psi, \gamma) = (J, S)$ and (J, S) is lattice order fuzzy soft set.

PROPOSITION 5. The extended intersection of two lattice order fuzzy soft sets (f, t) and (u, p) is a lattice (anti-lattice) ordered fuzzy soft set if $(f, t) \subseteq (u, p)$ or $(u, p) \subseteq (f, t)$.

Proof. It is the direct consequence of definition 10 and Proposition 4.

Let α , β be partially ordered sets. Then the partial order on $\alpha \times \beta$ is defined by dictionary order, that is for $(\nu_1, \mu_1) \leq (\nu_2, \mu_2)$ if and only if $\nu_1 \leq_\alpha \nu_2$ or if $\nu_1 = \nu_2$ and $\mu_1 \leq_\beta \mu_2$. So, we consider the partial order $\preccurlyeq$ on $\alpha \times \beta$ by defining, $(\nu_1, \mu_1) \preccurlyeq (\nu_2, \mu_2)$ if and only if $\nu_1 \leq_\alpha \nu_2$ and $\mu_1 \leq_\beta \mu_2$.

PROPOSITION 6. If (f, q) and (j, p) are lattice(anti-lattice) ordered fuzzy soft sets, then the basic intersection is also lattice(anti-lattice) ordered fuzzy soft set.

Proof. Let us consider (f, q) and (j, p). Let $(f, q) \wedge (j, p) = (\psi, \gamma)$, where $\gamma = q \times p$. Since $q, p \subseteq \Sigma$, so both q and p inherit a partial order from Σ and $\preccurlyeq$ order on $q \times p$. If $z_1 \leq_q z_2$ and $b_1 \leq_p b_2$, then we have $f(z_1) \subseteq f(z_2)$ and $j(b_1) \subseteq j(b_2)$ for all $z_1, z_2 \in q$ and for all $b_1, b_2 \in p$. Therefore, $(z_1, b_1) \preccurlyeq_\gamma (z_2, b_2)$. Now, $f(z_1) \cap j(b_1) \subseteq f(z_2) \cap j(b_2)$. This implies $\psi(z_1, b_1) \subseteq \psi(z_2, b_2)$, as required.

PROPOSITION 7. If (K, D) and (J, S) are lattice(anti-lattice) ordered fuzzy soft sets, then the basic union of these sets is a lattice(anti-lattice) ordered fuzzy soft set.

Proof. Proof follows from definition 8 and Proposition 6.

DEFINITION 18. If $\alpha, \beta \subseteq \Sigma$, then the partial order $\preccurlyeq$ on $\alpha \otimes \beta$ is induced by partial order on α and β as, for any $(a_1 \wedge b_1), (a_2 \wedge b_2) \in \alpha \otimes \beta$, $(a_1 \wedge b_1) \preccurlyeq (a_2 \wedge b_2)$ if and only if $a_1 \leq_\alpha a_2$ and $b_1 \leq_\beta b_2$.

PROPOSITION 8. Let (K, D) and (J, S) be a lattice(anti-lattice) ordered fuzzy soft set, then $(K, D) \cap_\wedge (J, S)$ is a lattice(anti-lattice) ordered fuzzy soft set.

Proof. Proof can be obtained from Definition 18 and Proposition 6.

PROPOSITION 9.

Let (λ, α) be a lattice ordered fuzzy soft set over the universe set. Then

1. $(\lambda, \alpha) \cap_R (\lambda, \alpha) = (\lambda, \alpha)$.

2. $(\lambda,\alpha) \cup_R (\lambda,\alpha) = (\lambda,\alpha)$.
3. $(\lambda,\alpha) \cap_R \Phi_\alpha = \Phi_\alpha$.
4. $(\lambda,\alpha) \cup_R \Phi_\alpha = (\lambda,\alpha)$.
5. $(\lambda,\alpha) \cap_R \chi_\alpha = (\lambda,\alpha)$.
6. $(\lambda,\alpha) \cup_R \chi_\alpha = \chi_\alpha$.

Proof. These results are straight forward, so we omit the proofs.

PROPOSITION 10. Let (λ,α), (ξ,β) and (ψ,γ) be any lattice ordered fuzzy soft sets over the same universe χ. Then

1. $(\lambda,\alpha) \cap_R ((\xi,\beta) \cap_R (\psi,\gamma)) = ((\lambda,\alpha) \cap_R (\xi,\beta)) \cap_R (\psi,\gamma)$.
2. $((\lambda,\alpha) \cup_R (\xi,\beta)) \cup_R (\psi,\gamma) = (\lambda,\alpha) \cup_R ((\xi,\beta)) \cup_R (\psi,\gamma))$.

Proof. Proof can be acquired from respective definitions.

Remark: From Proposition 2, it is clear that
$(\lambda,\alpha) \cap_{\mathcal{R}} ((\xi,\beta)) \cap_{\mathcal{R}} (\psi,\gamma)) = ((\lambda,\alpha) \cap_R (\xi,\beta)) \cap_R (\psi,\gamma)$
is a lattice ordered fuzzy soft set.

PROPOSITION 11. Let (λ,α) , (ξ,β) and (ψ,γ) be any lattice ordered fuzzy soft sets over the same universe χ. Then

1. $(\lambda,\alpha) \cup_R ((\xi,\beta)) \cap_R (\psi,\gamma)) = \big((\lambda,\alpha) \cup_R (\xi,\beta)\big) \cap_R \big((\lambda,\alpha) \cup_R (\psi,\gamma)\big)$.
2. $(\lambda,\alpha) \cap_R ((\xi,\beta)) \cup_R (\psi,\gamma)) = \big((\lambda,\alpha) \cap_R (\xi,\beta)\big) \cup_R \big((\lambda,\alpha) \cap_R (\psi,\gamma)\big)$.

Proof. (1) If $\alpha \cup (\beta \cap \gamma) = \Phi$ and $(\alpha \cup \beta) \cap (\alpha \cup \gamma) = \Phi$.
Then $(\lambda,\alpha) \cup_R ((\xi,\beta)) \cap_R (\psi,\gamma)) = ((\lambda,\alpha) \cup_R (\xi,\beta)) \cap_R ((\lambda,\alpha) \cup_R (\psi,\gamma))$ holds trivially.

Consider $(\lambda,\alpha) \cup_R ((\xi,\beta)) \cap_R (\psi,\gamma))$ and let $(\xi,\beta) \cap_R (\psi,\gamma) = (I,J)$, where $J = \beta \cap \gamma \neq \Phi$ and for all $e \in J$, $\xi(e) \wedge \psi(e) = I(e)$. Also let $(\lambda,\alpha) \cup_R (I,J) = (P,Q)$, where $Q = \alpha \cap J \neq \Phi$ and for $e_1 \in Q$, $\lambda(e_1) \vee I(e_1) = P(e_1)$.

Therefore, $\lambda(e_1) \vee I(e_1) = \lambda(e_1) \vee (\xi(e_1) \wedge \psi(e_1)) = P(e_1)$.

Now consider $((\lambda,\alpha) \cup_R (\xi,\beta)) \cap_R ((\lambda,\alpha) \cup_R (\psi,\gamma))$ and let $(\lambda,\alpha) \cup_R (\xi,\beta) = (S,T)$ and $T = \alpha \cap \beta \neq \Phi$ and for $e \in T, \lambda(e) \vee \xi \;(e) = S(e)$ together with $(\lambda,\alpha) \cup_R (\psi,\gamma) = (Y,\gamma)$, where $\gamma = \alpha \cap \gamma \neq \Phi$ and for all $e \in \gamma$, $\lambda(e) \vee \psi(e) = Y(e)$.

Now let $(S,T) \cap_R (Y,\gamma) = (U,V)$ where $V = T \cap \gamma \neq \Phi$ and for all $e_1 \in V$,
$S(e_1) \wedge Y(e_1) = U(e_1)$. Hence $S(e_1) \wedge Y(e_1) = \lambda(e_1) \vee \xi \;(e_1) \wedge \lambda(e_1) \vee \psi(e_1) = U(e_1)$
$S(e_1) \wedge Y(e_1) = \lambda(e_1) \vee (\xi \;(e_1) \wedge \psi(e_1)) = U(e_1)$.

Therefore,
$P(e_1) = U(e_1)$for all $e_1 \in$ or $(\lambda,\alpha) \cup_R ((\xi,\beta)) \cap_R (\psi,\gamma)) = ((\lambda,\alpha) \cup_R (\xi,\beta)) \cap_R ((\lambda,\alpha) \cup_R (\psi,\gamma))$

(2) The proof is more of a same to the proof of (1) which is not discussed here.

PROPOSITION 12. Complement of a lattice ordered fuzzy soft set is an anti-lattice ordered fuzzy soft set.

Proof. Let (λ, α) be a lattice ordered fuzzy soft set. Then $a_1 \leq_\alpha a_2$ implies $\lambda(a_1) \subseteq \lambda(a_2)$ $\Rightarrow$ $(\lambda(a_2))^c \subseteq (\lambda(a_1))^c$ for all $a_1, a_2 \in \alpha$ which implies that the complement of a lattice ordered fuzzy soft set is an anti-lattice ordered fuzzy soft set.

PROPOSITION 13. Let (λ, α) and (ξ, β) be two lattice ordered fuzzy soft sets over the same universe U, then

1. $((\lambda, \alpha) \cup_R (\xi, \beta))^c = (\lambda, \alpha)^c \cap_R (\xi, \beta)^c$
2. $((\lambda, \alpha) \cap_R (\xi, \beta))^c = (\lambda, \alpha)^c \cup_R (\xi, \beta)^c$
3. $((\lambda, \alpha) \vee (\xi, \beta))^c = (\lambda, \alpha)^c \wedge (\xi, \beta)^c$
4. $((\lambda, \alpha) \wedge (\xi, \beta))^c = (\lambda, \alpha)^c \vee (\xi, \beta)^c$

Proof. (1) Let $(\lambda, \alpha) \cup_R (\xi, \beta) = (\psi, \gamma)$, where $\psi(e) = \lambda(e) \vee \xi(e)$, for all $e \in \gamma = \alpha \cap \beta \neq \Phi$.
Since $((\lambda, \alpha) \cup_R (\xi, \beta))^c = (\psi, \gamma)^c$, then by definition:
$\psi(e)^c = 1 - [\lambda(e) \vee \xi(e)] = [1 - \lambda(e)] \wedge [1 - \xi(e)]$, for all $e \in \gamma$.
Now $(\lambda, \alpha)^c \cap_R (\xi, \beta)^c = (\lambda^c, \alpha) \cap_R (\xi^c, \beta) = (T, \gamma)$, where $\gamma = \alpha \cap \beta$.
So $T(e) = \lambda^c(e) \wedge \xi^c(e) = (1 - \lambda(e)) \wedge (1 - \xi(e)) = \psi(e)^c$
for all $e \in \gamma$. Hence $((\lambda, \alpha) \cup_R (\xi, \beta))^c = (\lambda, \alpha)^c \cap_R (\xi, \beta)^c$.

(2) Proof can be done in a similar fashion as in (1).

(3) Suppose $(\lambda, \alpha) \vee (\xi, \beta) = (\psi, \gamma)$, where $\gamma = \alpha \times \beta$. Therefore, $((\lambda, \alpha) \vee (\xi, \beta))^c = (\psi, \gamma)^c = (\psi^c, \gamma)$.
Now $(\lambda, \alpha)^c \wedge (\xi, \beta)^c = (\lambda^c, \alpha) \wedge (\xi^c, \beta) = (T, \gamma)$, where $T(e_1, e_2) = \lambda^c(e_1) \wedge \xi^c(e_2)$ when $(e_1, e_2) \in \alpha \times \beta$.
Let $(e_1, e_2) \in \gamma = \alpha \times \beta$. Then,
$\psi^c(e_1, e_2) = 1 - \psi(e_1, e_2) = 1 - [\lambda(e_1) \vee \xi(e_2)]$
$= [1 - \lambda(e_1)] \wedge [1 - \xi(e_2)]$
$= \lambda^c(e_1) \wedge \xi^c(e_2) = T(e_1, e_2)$.
Therefore, (ψ^c, γ) and $T(e_1, e_2)$ are same.
Hence, $((\lambda, \alpha) \vee (\xi, \beta))^c = (\lambda, \alpha)^c \wedge (\xi, \beta)^c$.

(4) A reader can prove this on the similar lines to (3).

4. Algebraic Structures of Lattice(anti-lattice) Ordered Fuzzy Soft Sets

In this very section algebraic structures of lattice ordered fuzzy soft sets are studied. This study is crucial for the application of lattice ordered fuzzy soft sets. For the universe χ and initial parameterΣ, we denote $\lambda\Theta oo(U)^{\alpha}$ as the collection of all lattice ordered fuzzy soft sets defined over χ with a fixed set of parameters α. Then keeping in view the results in the previous section we have the following proposition.

PROPOSITION 14.
1. $(\lambda\Theta oo(U)^{\alpha}, \cap_R)$ is a commutative idempotent monoid with identity χ_{α}.
2. $(\lambda\Theta oo(U)^{\alpha}, \cup_R)$ is a commutative idempotent monoid with identity Φ_{α}.
3. $(\lambda\Theta oo(U)^{\alpha}, \cup_R, \cap_R)$ is a commutative, idempotent semi ring.
4. $(\lambda\Theta oo(U)^{\alpha}, \cap_R, \cup_R)$ is a commutative, idempotent semi ring.
5. $(\lambda\, \Theta oo(U)^{\alpha}, \cap_R, \cup_R, {}^{c}, \chi_{\alpha}, \Phi_{\alpha})$ is a De Morgan's Lattice.
6. $(\lambda\Theta oo(U)^{\alpha}, \cup_R, \cap_R, {}^{c}, \Phi_{\alpha}, \chi_{\alpha})$ is a De Morgan's Lattice.

PROPOSITION 15. $(\lambda\Theta oo(U)^{\alpha}, \cap_{R,} \cup_R, {}^{c}, \chi_{\alpha}, \Phi_{\alpha})$ is a Kleene algebra.

Proof. From Proposition 14, $(\lambda\Theta oo(U)^{\alpha}, \cap_{R,} \cup_R, {}^{c}, \chi_{\alpha}, \Phi_{\alpha})$ is a De Morgan algebra.
Now, for any $(\lambda, \alpha), (\xi, \alpha) \in \lambda\Theta oo(U)^{\alpha}$, if we have
$(\lambda, \alpha) \cap_R (\lambda, \alpha)^c \tilde{\supseteq} (\xi, \alpha) \cup_R (\xi, \alpha)^c$
Then there exists some $c \in \alpha$ such that $\lambda(c) \cap \lambda^c(c) \supset \xi(c) \cup \xi^c(c)$ and there exist some $t \in \chi$ such that

$$((\lambda(c) \cap \lambda^c(c))(t) > ((\xi(c) \cup \xi^c(c)))(t)$$
$$\text{or } (\lambda(c) \wedge \lambda^c(c))(t) > (\xi(c) \vee \xi^c(c))(t)$$
$$\text{or } ((\lambda(c))(t) \wedge (\lambda^c(c))(t) > (\xi(c))(t) \vee \xi^c(c))(t).$$

Now suppose that $(\lambda(c))(t) = \alpha$, where $\alpha \in [0,1]$. Then $(\lambda^c(c))(t) = 1 - \alpha$.
Case 1: if $1 - \alpha \geq \alpha$, then:

$$\alpha \wedge (1 - \alpha) > (\xi(c))(t) \vee \xi^c(c))(t)$$
$$\alpha > (\xi(c))(t) \vee \xi^c(c))(t)$$

that is $\alpha > (\xi(c))(t)$ and $\alpha > \xi^c(c))(t)$.
Now $1 - \alpha < 1 - (\xi(c))(t)$ that is $1 - \alpha < \xi^c(c))(t)$ and $\xi^c(c))(t) < \alpha$. So, contradiction occurs because of $1 - \alpha < \alpha$.

Case 2: if $\alpha \geq 1 - \alpha$, then by a similar argument as in case 1, we get the contradiction $\alpha < 1 - \alpha$.
Hence $(\lambda, \alpha) \cap_R (\lambda, \alpha)^c \tilde{\subseteq} (\xi, \alpha) \cup_R (\xi, \alpha)^c$.
Therefore, $(\lambda\ \Theta oo(U)_{\alpha}, \cap_{R,} \cup_R, {}^{c}, \Phi_{\alpha}, \chi_{\alpha})$ is a Kleene algebra.

5. Applications of Lattice Ordered Fuzzy Soft Sets

Lattice ordered fuzzy soft sets are also imperative in several world higher cognitive process issues. In the following example, a decision making problem is being considered, where data can be represented by lattice ordered fuzzy soft sets.

EXAMPLE 2. An Agricultural Research Institute is interested in acquiring an experimental land. The purpose behind is to investigate the growth of different plants and crops. Five different options of land at different locations are available. Two experts were sent to point out the suitable and best option of land. Both have expertise in different aspects related to agricultural land. Let $\chi = \{l_1, l_2, l_3, l_4, l_5\}$ be the set of lands (universe set) and let $\Sigma = \{e_1$(Price), e_2(soil nature), e_3(availability of water), e_4(weather conditions), e_5(fertility), e_6(easy approach), e_7(quantity of salts in soil)$\}$ be the set of parameters to select the land, while selecting a land there is an order to choose on set Σ which is shown in figure 2.

Order among elements of Σ.

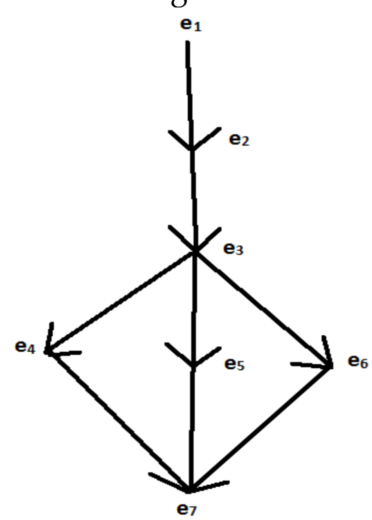


Fig.2

Suppose expert 1 recommends a land by considering parameter set $\alpha = \{e_1, e_3, e_4, e_6\}$ and his/her fellow takes care of parameter set $\beta = \{e_2, e_5, e_7\}$. As order on α and β is inherited from Σ . Let (λ, α) and (ξ, β) be lattice ordered fuzzy soft sets as shown in Table 2 and 3.

Soft set (λ, α)

(λ, α)	e1	e3	e4	e_6
l_1	0.3	0.4	0.6	0.7
l_2	0.2	0.5	0.5	0.6
l_3	0.2	0.5	0.6	0.8
l_4	0.4	0.4	0.6	0.9
l_5	0.2	0.6	0.8	0.9

Table 2

Soft set (ξ, β)

(ξ, β)	e_2	e_5	e_7
l_1	0.3	0.7	0.8
l_2	0.3	0.5	0.6
l_3	0.3	0.8	0.9

l_4	0.4	0.9	0.9
l_5	0.5	0.9	0.9

Table 3

The opinion of both members carry same weight. The combination of both opinions will help to select suitable choice. In order to do so, consider $(\lambda, \alpha) \cap_{\wedge} (\xi, \beta) = (I, D)$ of lattice ordered fuzzy soft set is taken which is shown in table 4, where $D = \alpha \otimes \beta$ and order on D is shown in figure 3.

$Order\ among\ elements\ of\ D$

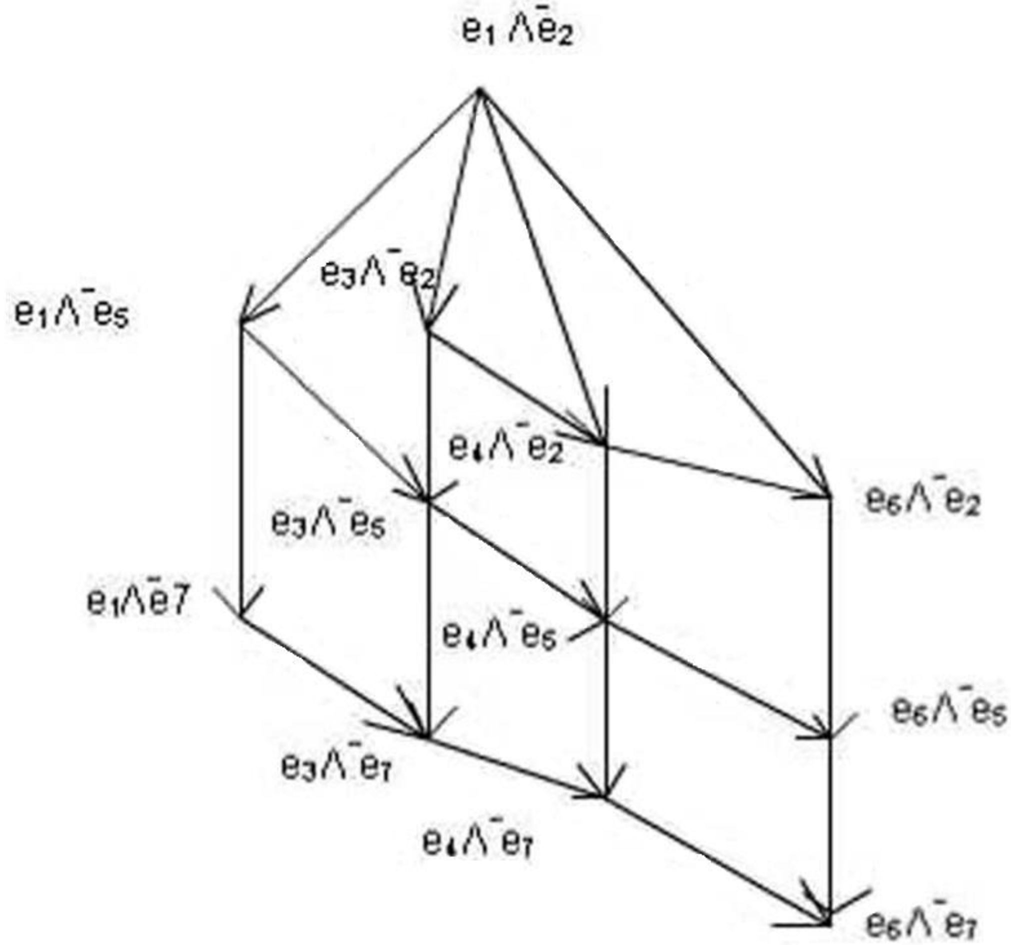


Figure 3

Soft set (I, D)

(I, D)	l_1	l_2	l_3	l_4	l_5
$(e_1 \wedge^- e_2)$	0.3	0.3	0.3	0.4	0.5
$(e_1 \wedge^- e_5)$	0.7	0.5	0.8	0.9	0.9
$(e_1 \wedge^- e_7)$	0.8	0.6	0.9	0.9	0.9
$(e_3 \wedge^- e_2)$	0.4	0.5	0.5	0.4	0.6
$(e_3 \wedge^- e_5)$	0.7	0.5	0.8	0.9	0.9
$(e_3 \wedge^- e_7)$	0.8	0.6	0.9	0.9	0.9
$(e_4 \wedge^- e_2)$	0.6	0.5	0.6	0.6	0.8
$(e_4 \wedge^- e_5)$	0.6	0.5	0.8	0.9	0.9
$(e_4 \wedge^- e_7)$	0.8	0.6	0.9	0.9	0.9
$(e_6 \wedge^- e_2)$	0.7	0.6	0.8	0.9	0.9
$(e_6 \overline{\wedge} e_5)$	0.7	0.6	0.8	0.9	0.9
$(e_6 \wedge^- e_7)$	0.8	0.6	0.9	0.9	0.9

Table 4

Then (I, D) is lattice ordered fuzzy soft set. In Table 5 all the membership grades are sum up for all parameters,

	l_1	l_2	l_3	l_4	l_5
$\sum$	7.9	6.4	9.0	9.5	10.0

Table 5

The selection of land will be judged by maximum membership grades in Table 5. Thus the ranking of land is given in Table 6.

Land	Score in Table 5	Rank
l_1	7.9	4th
l_2	6.4	5th
l_3	9.0	3rd
l_4	9.5	2nd
l_5	10.0	1st

Table 6

Conclusion: Sometimes linguistic terms have a particular order. Soft sets having a particular order on the entries of parameters set are very useful to represent such data. In this paper, idea of lattice (anti-lattice) ordered fuzzy soft sets is studied. This idea can also be very useful in higher cognitive process issues, wherever sure order exists in linguistic terms.

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