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Extension of GTS through hereditary class

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Abstract

In literature there are the notions of generalized topology (in short GT) and hereditary class. Both these notions have been studied extensively in different directions. Most of these studies evolved around considering a generalized topological space (in short GTS) endowed with certain hereditary class, and subsequently the nature of the resultant space is studied. By this, mathematicians intend to measure the quantum of deviations, the resultant space receives in respect of its basic properties. In our work we are deviating from this course of study. We consider an arbitrary GTS and certain collection of hereditary classes on this space and next we lift this collection to a GTS and the initial GTS will be embedded to this newly formulated space through certain well defined embedding map. Further we study the compactness property of this extension space. In course of this study we introduce certain notions and corresponding results for the theories of GTS as well as of hereditary class.

Keywords: Generalized topology, closed base for GT, hereditary class, embedding, extension, generalized compactness.

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1. Introduction

From literature, it is observed that mathematicians find it quite interesting to extend a given space to another one in such a way that the latter one enjoys certain property which was missing in the former one. Compactification of topological space is such an extension. Till now there exist different types of compactifications, e.g., One-point compactification, Stone-Čech compactification and so many.

In 2002, Á. Császár [2] introduced the concept of generalized topology (in short, GT), which is an weaker version of topology. More accurately, the study was initiated in [1] by Császár himself. However, the article was about monotonic mappings and generalized topology was nowhere there. But in [2], generalized topology emerged as a particular case of this notion in [1]. Subsequently, it has been observed that the collections of semiopen, preopoen, δ-open, θ-open, regular open and a lot many other collections of sets on a topological space form generalized topologies on the underlying set. Therefore, beginning as mere a

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weaker structure than topology, GT turns out to be a powerful unification of a large collections of sets in a topological space. So, quite naturally, GT is being studied thoroughly by a large number of mathematicians (e.g., [4], [3], [5], [8], [9], [17], [15]) till now.

Again, the study of topological spaces with additional structures dates back to sixties. In 1960, R. Vaidyanathaswamy [20] introduced a topological space with an additional structure, viz. an ideal, on it. Much later, D. Janković and T. R. Hamlett studied this concept extensively in their 1990's article [12]. Following this paper there has been a thorough study of ideal topology by different mathematicians till now and it is still continuing. On the same time some mathematicians started to generalize the concept of ideal topology by different ways. Some considered generalized topology (GT) instead of topology, whereas some replaced ideal with a somewhat weaker structure, called hereditary class. Further, some have considered both these generalizations simultaneously. In our study, we go along the last approach, with a typical aim in mind. We are carrying forward the work of [16] in generalized topological settings. We begin with a GT and consider a collection of hereditary classes on that space to form another GT on this collection so that the former space can be embedded into the latter.

In the next two consecutive sections we will recapitulate the necessary literature concerning generalized topology and hereditary class. Not only that, here we will also introduce certain new notions and corresponding results to develop the theories of generelized topology and hereditary class to some new extent. Afterwards we will attend our main goal to get a typical extension of a generalized topological space. And finally in the very last section we will discuss about the compactness property of this extension space.

2. Prerequisites & Preliminaries of *GT*

In this section we assemble certain basic facts about generalized topology from the existing literature and also study some new aspects to enable us for the proceedings afterwards. Firstly, we go through the existing literature.

Definition 2.1. [2] A non-empty collection μ of the power set of a non-empty set X is said to form a generalized topology (in short, GT) on X if

1. $\phi \in \mu$.

2. $U_\alpha \in \mu, \forall \alpha \in \Lambda \implies \bigcup_{\alpha \in \Lambda} U_\alpha \in \mu$.

The pair (X, μ) is called a generalized topological space (in short, GTS) and the members of μ are called generalized open (or μ-open) sets.

Definition 2.2. [5] The complement of a generalized open set is called generalized closed. If μ is the concerned GT, then generalized open and generalized closed sets are also termed as μ-open and μ-closed respectively.

Remark 2.3. It is apparent from the above two definitions that the whole set is always generalized closed and arbitrary intersection of generalized closed sets is also generalized closed.

Definition 2.4. [5] For a non-empty subset A of a GTS (X, μ),

(i) μ-closure of A, denoted by $c_\mu(A)$, is defined to be the least μ-closed set containing A i.e., the intersection of all μ-closed sets containing A.

(ii) μ-interior of A, denoted by $i_\mu(A)$, is defined to be the greatest μ-open set contained in A i.e., the union of all μ-open sets contained in A.

Result 2.5. [5] Let, A be a non-empty subset of a GTS (X, μ). Then,

(i) $c_\mu(A) = \{x \in X : A \cap G \neq \emptyset, \forall G \in \mu \text{ with } x \in G\}$.

(ii) $i_\mu(A) = \{x \in X : \exists\, G \in \mu \text{ such that } x \in G \subseteq A\}$.

Result 2.6. [5] Let, (X, μ) be a GTS and A be a non-empty subset of X. Then,

(i) A is μ-closed iff $A = c_\mu(A)$.

(ii) A is μ-open iff $A = i_\mu(A)$.

Definition 2.7. [3] A GT is said to be a strong GT (in another terminology, supratopology [11]) if the concerning whole space is also a member of the GT.

Definition 2.8. [6] Let (X, μ) be a GTS. Then $\mu' \subseteq \mu$ is said to be a base for μ, in other words, a *μ-base* or a *μ-open base*, if whenever $G \in \mu$ and $x \in G$, then $\exists\, B \in \mu'$ such that $x \in B \subseteq G$.

Proposition 2.9. [6] A subcollection β of $\exp(X)$ in a non-empty set X forms a base for some GT on X if $\phi \in \beta$.

The collection consisting of arbitrary unions of members of β forms the GT and it is said to be generated by β.

Proposition 2.10. [13] For a non-empty set X, $\mu' \subseteq \exp(X)$ forms a *base* for some strong generalized topology on X if and only if $X = \bigcup\limits_{B \in \mu'} B$.

Definition 2.11. [2] Let, (X, μ) and $(Y, \mu'$ be two GTSs. Then a function $f : X \to Y$ is said to be (μ, μ')-continuous (or simply, g-continuous, where μ and μ' are known) if $G \in \mu' \implies f^{-1}(G) \in \mu$.

This notion of generalized continuity i.e., g-continuity has an equivalent form in terms of the points of the space. We have the next formulation in this regard.

Theorem 2.12. [2] f will be (μ, μ')-continuous if for each $x \in X$ and $G' \in \mu'$ with $f(x) \in G'$, $\exists G \in \mu$ such that $x \in G$ and $f(G) \subseteq G'$.

Theorem 2.13. [14] For a subset Y of a GTS (X, μ), the collection $\{G \cap Y : G \in \mu\}$ forms a GT on Y.

This GT is denoted by μ_Y and is called the subspace generalized topology on Y. Also Y together with μ_Y, i.e., (Y, μ_Y) is called a subspace of X.

Definition 2.14. [4] A GTS (X, μ) is said to be

1. μ-T_0 if given $x, y \in X$, with $x \neq y$, $\exists\, V \in \mu$ such that either $x \in V$ or $y \in V$, but not the both.
2. μ-T_1 if given $x, y \in X$, with $x \neq y$, $\exists\, U, V \in \mu$ such that $x \in U \setminus V$ and $y \in V \setminus U$.

Now we move on to introduce certain aspects of GT, which are very basic in nature. We further discuss certain related results.

Definition 2.15. Let (X, μ) be a GTS and $\mathcal{F}$ be the collection of all *μ-closed* sets in X. Then $\mathcal{F}' \subseteq \mathcal{F}$ is said to form a *base* for $\mathcal{F}$, in other words, a *μ-closed base*, if whenever $F \in \mathcal{F}$ and $x \notin F$, then $\exists\, C \in \mathcal{F}'$ such that $x \notin F \subseteq C$.

Proposition 2.16. A subcollection $\mathcal{F}$ of $\exp(X)$ of a non-empty set X forms a base for the generalized closed sets of a GT on X if $X \in \mathcal{F}$.

The collection consisting of arbitrary intersections of members of $\mathcal{F}$ is the collection of generalized closed sets for the induced GT.

Proof. The result is dual to the corresponding result for generalized open base. Hence the proof follows similarly, by maintaining the dual character of generalized open and generalized closed sets.

Remark 2.17. From the corresponding definitions it is vividly clear that any non-empty subcollection of $\exp(X)$ in a non-empty set X induces a GT on the under-lying set, by forming a generalized open or closed base. It may also be noted that the same collection can be used to get a generalized open base, as well a generalized closed base, but surely the two will induce different GT. For the first case, the empty set ϕ is to be attached together with the collection, whereas, in the second case, the whole set X is to be attached.

Proposition 2.18. For a non-empty set X, $\mathcal{F}' \subseteq \exp(X)$ forms a closed base for some strong generalized topology on X if and only if $\bigcap_{F\in\mathcal{F}'} F = \phi$.

Proof. The result is dual to that of Proposition 2.10.

Definition 2.19. Let (X,μ) be a GTS and $x \in X$. Then a subset A of X is said to be a μ-neighbourhood or, *μ-nbd* of x if there exists $G \in \mu$ such that $x \in G \subseteq A$.

Definition 2.20. Let (X,μ) be a GTS and $x \in X$. Then $\mu' \subseteq \mu$ is said to be a local base at x, if whenever $x \in G \in \mu$, then $\exists\ V \in \mu'$ such that $x \in V \subseteq G$.

Now we note certain further formulations and properties of interior and closure operators in generalized topological space.

Proposition 2.21. Let (X,μ) be a GTS. Then for a subset A of X,

(a) $x \in c_\mu(A)$ if and only if every *μ-nbd* of x intersects A, where $x \in X$.
(b) $x \in i_\mu(A)$ if and only if $\exists$ a *μ-nbd* of x, contained in A, where $x \in X$.
(c) $c_\mu(A)$ is *μ-closed*.
(d) $i_\mu(A)$ is *μ-open*.

Proof. From the definition of *μ-nbd* it is obvious that every *μ-open* set is a *μ-nbd* of its members and also between any point $x(\in X)$ and a *μ-nbd* of the point, there exist a *μ-open* set containing the point and contained in the *μ-nbd*. From these two facts the first two result follows.

For the third result, let us consider $x \in X \setminus c_\mu(A)$. Then $\exists\ G \in \mu$ such that $x \in G$ and $G \cap A = \phi$. If $\exists$ $y \in G \cap c_\mu(A)$ then it turns out that G is a *μ-open* set containing y and $G \cap A = \phi$. It's a contradiction to the fact that $y \in c_\mu(A)$. Thus, $G \cap c_\mu(A) = \phi$ and hence $x \notin c_\mu(c_\mu(A))$. So, $c_\mu(A)$ is closed.

The final assertion can also be derived in similar way.
Now we pay heed to formulate certain functional aspects and relevant results on generalized topological space.

Definition 2.22. A function $f : X \to Y$, where (X,μ) and (Y,μ') are two generalized topological spaces, is said to be (μ,μ')-open (or simply, generalized open, when μ and μ' are known) if $G \in \mu \implies f(G) \in \mu'$.

Definition 2.23. A function $f : X \to Y$, where (X,μ) and (Y,μ') are two generalized topological spaces, is said to be (μ,μ')-closed (or simply, generalized closed, when μ and μ' are known) if K is μ-closed implies $f(K)$ is μ'-closed.

Theorem 2.24. For a bigective mapping $f : X \to Y$, where (X,μ) and (Y,μ') are two generalized topological spaces, f is (μ,μ')-open iff it is (μ,μ')-closed.

Proof. The result follows simply from the fact that for any subset A of X, $f(X \setminus A) = Y \setminus f(A)$, as f is bijective.

Theorem 2.25. A function $f : X \to Y$, where (X,μ) and (Y,μ') are two generalized topological spaces, is (μ,μ')-closed iff for any $A \subseteq X$, $c_{\mu'}(f(A)) \subseteq f(c_\mu(A))$.

Proof. First assume f to be (μ, μ')-closed and consider $A \subseteq X$. Then $A \subseteq c_\mu(A) \implies f(A) \subseteq f(c_\mu(A))$. Now f being (μ, μ')-closed, $f(c_\mu(A))$ is μ'-closed. Then from the definition of μ-closure it follows that $c_{\mu'}(f(A)) \subseteq f(c_\mu(A))$.

Conversely assume that the condition holds and consider $A \subseteq X$ to be μ-closed. So $A = c_\mu(A)$ and $c_{\mu'}(f(A)) \subseteq f(c_\mu(A))$ i.e., $c_{\mu'}(f(A)) \subseteq f(A)$. Again it is quite obvious that $f(A) \subseteq c_{\mu'}(f(A))$. Thus $f(A) = c_{\mu'}(f(A))$ and hence $f : X \to Y$ is (μ, μ')-closed.

Theorem 2.26. For a function $f : X \to Y$, where (X, μ) and (Y, μ') are two generalized topological spaces, the following are equivalent:

(i) f is (μ, μ')-continuous.

(ii) $f^{-1}(K)$ is μ-closed, for all μ'-closed sets K.

(iii) $\forall A \subseteq X$, $f(c_\mu(A)) \subseteq c_{\mu'}(f(A))$.

Proof. [(i) $\implies$ (ii)]: Let f be (μ, μ')-continuous and K be μ'-closed. So, $(Y \setminus K) \in \mu'$. Then f being (μ, μ')-continuous, $f^{-1}(Y \setminus K) \in \mu$. Now it is easy to verify that $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$. Thus $(X \setminus f^{-1}(K)) \in \mu$ and in turn $f^{-1}(K)$ is μ-closed.

[(ii) $\implies$ (iii)]: Assume that the condition (*ii*) holds and consider $A \subseteq X$. Now clearly $f(A) \subseteq c_{\mu'}(f(A))$. So $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(c_{\mu'}(f(A)))$. Again by the assumed condition it follows that $f^{-1}(c_{\mu'}(f(A)))$ is μ-closed. So from the definition of μ-closure it follows that $c_\mu(A) \subseteq f^{-1}(c_{\mu'}(f(A))) \implies f(c_\mu(A)) \subseteq f(f^{-1}(c_{\mu'}(f(A)))) \subseteq c_{\mu'}(f(A))$. Thus the result holds.

[(iii) $\implies$ (i)]: Suppose that the condition in (*iii*) holds and G be μ'-open. Now it can be verified easily that $(X \setminus f^{-1}(G)) = f^{-1}(Y \setminus G)$. Then by the assumed condition we have, $f(c_\mu(X \setminus f^{-1}(G))) \subseteq c_{\mu'}(f(X \setminus f^{-1}(G))) = c_{\mu'}(f(f^{-1}(Y \setminus G))) \subseteq c_{\mu'}(Y \setminus G) = Y \setminus G$, as $G \in \mu'$. So, $c_\mu(X \setminus f^{-1}(G)) \subseteq f^{-1}(Y \setminus G)$ i.e., $c_\mu(X \setminus f^{-1}(G)) \subseteq (X \setminus f^{-1}(G))$. Thus in turn $c_\mu(X \setminus f^{-1}(G)) = X \setminus f^{-1}(G)$ and hence $f^{-1}(G)$ is μ-open.

Remark 2.27. The condition (*iii*) of the above theorem can be replaced with the condition $c_{\mu'}(f(c_\mu(A))) = c_{\mu'}(f(A))$, $\forall A \subseteq X$.

Proof. As $c_{\mu'}(f(c_\mu(A)))$ is the least μ'-closed set containing $f(c_\mu(A))$, the result quite obviously.

Definition 2.28. For two generalized topological spaces (X, μ) and (Y, μ'), a function $f : X \to Y$ is called g-homeomorphism if f is bijective, (μ, μ')-open and (μ, μ')-continuous.

Theorem 2.29. Let Y be a subspace of a generalized topological space (X, μ). Then $c_{\mu_Y}(A) = c_\mu(A) \cap Y$, for any subset A of Y.

Proof. Consider Y to be a subspace of (X, μ) and $A \subseteq Y$. Now let $x \in c_{\mu_Y}(A)$ and $G \in \mu$ such that $x \in G$. Then $x \in G \cap Y$ and by the Theorem 2.13, $(G \cap Y) \in \mu_Y$. Therefore as $x \in c_{\mu_Y}(A)$, $(G \cap Y) \cap A \neq \phi$, So $G \cap A \neq \phi$. Thus $x \in c_\mu(A)$ and hence $c_{\mu_Y}(A) \subseteq c_\mu(A)$. Again by default $c_{\mu_Y}(A) \subseteq Y$. Thus $c_{\mu_Y}(A) \subseteq (c_\mu(A) \cap Y)$.

Now for the reverse assertion take $x \in (c_\mu(A) \cap Y)$ and $G_Y \in \mu_Y$ such that $x \in G_Y$. Now as $G_Y \in \mu_Y$, $\exists\, G \in \mu$ such that $G_Y = G \cap Y$. Then $G \cap A \neq \phi$, as $x \in c_\mu(A)$. Again obviously $(G \cap A) \subseteq Y$. Thus $(G \cap A) \cap Y \neq \phi$, i.e., $(G \cap Y) \cap A \neq \phi$, i.e., $G_Y \cap A \neq \phi$. Thus $x \in c_{\mu_Y}(A)$ and hence $(c_\mu(A) \cap Y) \subseteq c_{\mu_Y}(A)$. Thus the result follows.

Theorem 2.30. An one-to-one function $f : X \to f(X)(\subseteq Y)$, where (X, μ) and (Y, μ') are two generalized topological spaces, is a g-homeomorphism iff for any $A \subseteq X$, $f(c_\mu(A)) = c_{\mu'}(f(A)) \bigcap f(X)$.

Proof. The result follows from the Theorems 2.24, 2.25, 2.26 and 2.29.

Here we must note that all the foregoing notions and results can be studied by considering generalized open or generalized closed basic members instead of generalized open and closed sets respectively. With this note we abandon this discussion right here, as it would suffice our future requirements.

3. Hereditary Classes on *GTS*

In this section, firstly we recall what a hereditary class is and then directly pay heed to certain special hereditary classes on generalized topological spaces. These will be required for the construction of the proposed extension.

Definition 3.1. [7] A non-empty subcollection $\mathcal{H}$ of the power set of a non-empty set X is said to be a hereditary class if $A \in \mathcal{H}$ and $B \subseteq A \implies B \in \mathcal{H}$.

Lemma 3.2. In a *GTS* (X, μ), for each $x \in X$,

(a) The collection $\mathcal{H}_{c_\mu}(x) = \{A \subseteq X : x \notin c_\mu(A)\}$ forms a hereditary class on X.
(b) The collection $\mathcal{H}_{\eta_\mu}(x) = \{A \subseteq X : A$ is not a μ-neighbourhood of $x\}$ forms a hereditary class on X.

Proof. Straightforward.

We now intend to define the following terminology.

Definition 3.3. In the above lemma,

(i) the hereditary class $\mathcal{H}_{c_\mu}(x)$ is termed as the adherence free hereditary class at x.

(ii) the hereditary class $\mathcal{H}_{\eta_\mu}(x)$ is termed as the nbd complement hereditary class at x.

Result 3.4. Let $\mathcal{H}$ be a hereditary class on a non-empty set X. Then the collection, $\mathcal{H}_d = \{A \subseteq X : (X \setminus A) \notin \mathcal{H}\}$ also forms a hereditary class on X.

Proof. It's quite obvious.

Definition 3.5. In the above result, $\mathcal{H}_d$ is called the dual of $\mathcal{H}$.

Example 3.6. In a *GTS* (X, μ), for a particular $x \in X$, the corresponding adherence free hereditary class and the nbd complement hereditary class are dual of each other. This can be easily verified from the corresponding descriptions of these classes.

Definition 3.7. A hereditary class $\mathcal{H}$ on a *GTS* (X, μ) is said to be closure preserving if $A \in \mathcal{H} \implies c_\mu(A) \in \mathcal{H}$.

Theorem 3.8. Every adherence free hereditary class in a *GTS* is closure preserving.

Proof. Follows from the relevant definitions.

Remark 3.9. The converse of the above theorem does not hold. Indeed, consider the space $(\mathbb{R}, \tau)$, where τ is the usual topology on $\mathbb{R}$. Then τ is certainly a *GT* on $\mathbb{R}$ and also $(\mathbb{R}, \tau)$ is τ-T_1. Then consider the collection $\mathcal{H} = \{A \subseteq \mathbb{R} : A \subseteq [0, 1]\}$. It can easily be shown that $\mathcal{H}$ is a closure preserving hereditary class on $\mathbb{R}$. It is quite obvious that $\mathcal{H} \neq \mathcal{H}_{c_\mu}(x)$, for any $x \in [0, 1]$. Also for no $x \in (X \setminus [0, 1])$, $\mathcal{H} = \mathcal{H}_{c_\mu}(x)$ holds, as $(\mathbb{R}, \tau)$ being τ-T_1, $\exists\, x'(\neq x) \in (X \setminus [0, 1])$ such that $\{x'\} \in \mathcal{H}_{c_\mu}(x)$, but obviously, $\{x'\} \notin \mathcal{H}$.

Theorem 3.10. A *GTS* (X, μ) is μ-T_0 iff $H_{c_\mu}(x) \neq H_{c_\mu}(y)$, for each pair of distinct $x, y \in X$.

Proof. Let, (X, μ) be a μ-T_0 *GTS*. Then $\exists\, V \in \mu$ such that either $x \in V$ or $y \in V$, but not the both. Thus in turn, either $x \notin c_\mu\{y\}$ or $y \notin c_\mu\{x\}$ or may be both. So, without loosing any generality we may assume that $x \notin c_\mu\{y\}$. Hence, $\{y\} \in H_{c_\mu}(x)$, whereas it is quite obvious that $\{y\} \notin H_{c_\mu}(y)$. Thus, $H_{c_\mu}(x) \neq H_{c_\mu}(y)$.

Conversely suppose that, the given condition holds and x and y are two distinct points of X. Then it follows from the given condition that $\exists\, A \subseteq X$ such that $A \notin H_{c_\mu}(x)$, but $A \in H_{c_\mu}(y)$, i.e., $x \in c_\mu(A)$ and $y \notin c_\mu(A)$. Hence, $\exists\, V \in \mu$ such that $y \in V$ and $V \cap A = \phi$. Again, as $x \in c_\mu(A)$, $x \notin V$. Thus (X, μ) is μ-T_0.

Definition 3.11. [10] A hereditary class $\mathcal{I}$ on a non-empty set X is called an ideal if $A, B \in \mathcal{I}$ implies $(A \cup B) \in \mathcal{I}$.

From the respective definitions it is clear that an ideal on a non-empty set is in turn a hereditary class on the same set. So, similar results (refer to [16]) are obtained if we replace *GT* by topology and hereditary class by ideal in Lemma 3.2 to Theorem 3.10. At this point it must be noted that if in the above results, $\mathcal{H}$ is taken to be precisely a hereditary class but not an ideal, then all the other derived hereditary classes are also mere hereditary classes and not ideals.

4. Extension of *GTS*

We begin this section with the introduction of extension of generalized topological spaces. Here we go accordingly as it was in case of topology.

Definition 4.1. An extension of a *GTS* (X, μ) is another *GTS* (Y, μ') together with a function $f : X \to Y$, such that $f : X \to f(X)(\subseteq Y)$ is g-homeomorphism and $c_{\mu'}(f(X)) = Y$.

Here f is termed as the extension function.

Definition 4.2. Two extensions (Y, μ_Y) and (Z, μ_Z) of a *GTS* (X, μ), via the functions f_Y and f_Z respectively, are said to be equivalent if $\exists$ a g-homeomorphism $f : Y \to Z$ such that $f \circ f_Y = f_Z$.

Definition 4.3. Let, (Y, μ') be an extension of a *GTS* (X, μ) via the function f, and y be an arbitrary point of Y. Then the strength of y, denoted by $S(y)$, is defined as

$$S(y) = \{A \subseteq X : y \notin c_{\mu'}(f(A))\}.$$

The collection, $S(Y) = \{S(y) : y \in Y\}$ is called the strength system of this extension.

Result 4.4. Let, (Y, μ') be an extension of a *GTS* (X, μ) via the function f. Then $S(y)$ is a closure preserving hereditary class on X, for each $y \in Y$.

Proof. It follows from Theorem 2.30.

Theorem 4.5. Let, (Y, μ') be an extension of a *GTS* (X, μ) via the function f. Then $S(f(x)) = \mathcal{H}_{c_\mu}(x)$.

Proof. Indeed, $S(f(x)) = \{A \subseteq X : f(x) \notin c_{\mu'}(f(A))\} = \{A \subseteq X : f(x) \notin c_{\mu'}(f(A)) \cap f(X)\} = \{A \subseteq X : f(x) \notin f(c_\mu(A))\} = \{A \subseteq X : x \notin c_\mu(A)\} = \mathcal{H}_{c_\mu}(x)$.

Theorem 4.6. If the strengths of different points of an extension space of a generalized topological space are different, then the extension space is T_0 with respect to the corresponding *GT*.

Proof. Let (Y, μ') be an extension of a *GTS* (X, μ) via the function f and the strengths of different points of Y be different. Take $y, z \in Y$ such that $y \neq z$ and hence $S(y) \neq S(z)$. Then $\exists\, A \subseteq X$ such that $y \notin c_{\mu'}(f(A))$ and $z \in c_{\mu'}(f(A))$. So $\exists\, V \in \mu'$ such that $y \in V$ and $V \cap f(A) = \phi$. So $z \notin V$, as $z \in c_{\mu'}(f(A))$. Hence, Y is μ'-T_0.

Theorem 4.7. Equivalent extensions of a generalized topological space have identical strength systems.

Proof. Let (Y_1, μ_1) and (Y_2, μ_2) be two extensions of a *GTS* (X, μ) via the functions f_1 and f_2 respectively and also let the extensions be equivalent through the g-homeomorphism $f : Y_1 \to Y_2$. Then $f \circ f_1 = f_2$. Take $y \in Y_1$. Then $A \in S(y) \Leftrightarrow y \notin c_{\mu_1}(f_1(A)) \Leftrightarrow f(y) \notin f(c_{\mu_1}(f_1(A))) \Leftrightarrow f(y) \notin c_{\mu_2} f((f_1(A))) \Leftrightarrow f(y) \notin c_{\mu_2}(f_2(A)) \Leftrightarrow A \in S(f(y))$. Thus, $S(y) = S(f(y))$,and hence the result follows.

With all these notions of extensions and their equivalence, we now proceed to construct certain particular extension of a given generalized topological space.

Theorem 4.8. Let (X,μ) be a μ-T_0 GTS and consider a collection $\mathfrak{X}$ of closure preserving hereditary classes on X, containing all the adherence free hereditary classes and also no member of $\mathfrak{X}$ contains X. For each $A \subseteq X$, define $A^{c_\mu} = \{\mathcal{H} \in \mathfrak{X} : A \notin \mathcal{H}\}$. Then the collection $\{A^{c_\mu} : A \subseteq X\}$ forms a base for generalized closed sets for some strong GT, say $\mu_{\mathfrak{X}}$, on $\mathfrak{X}$. Then $(\mathfrak{X},\mu_{\mathfrak{X}})$ forms an extension of (X,μ), via some suitably defined function.

Proof. Following Proposition 2.18, the construction of $(\mathfrak{X},\mu_{\mathfrak{X}})$ is certainly justified. Let us now define a function $f : X \to \mathfrak{X}$ as $f(x) = \mathcal{H}_{c_\mu}(x)$. Then (X,μ) being μ-T_0, f is a one-to-one function (by Theorem 3.10).

Now we will prove that for any $A \subseteq X$, $f(c_\mu(A)) = (c_{\mu_{\mathfrak{X}}}(f(A))) \bigcap f(X)$. Now $A \subseteq X$. So $c_\mu(A) \subseteq X \implies f(c_\mu(A)) \subseteq f(X)$. Now take $x \in c_\mu(A)$ and $B \subseteq X$ such that $f(A) \subseteq B^{c_\mu}$. Then $\mathcal{H}_{c_\mu}(x) = f(x) \in f(c_\mu(A))$. Again $f(A) = \{\mathcal{H}_{c_\mu}(a) : a \in A\}$. So, as $f(A) \subseteq B^{c_\mu}$, $\mathcal{H}_{c_\mu}(a) \in B^{c_\mu}$, $\forall\, a \in A$ i.e., $B \notin \mathcal{H}_{c_\mu}(a)$, $\forall\, a \in A$. Hence $a \in c_\mu(B)$, $\forall\, a \in A \implies A \subseteq c_\mu(B) \implies c_\mu(A) \subseteq c_\mu(B)$. Thus $x \in c_\mu(B)$ so that $B \notin \mathcal{H}_{c_\mu}(x)$ and hence $\mathcal{H}_{c_\mu}(x) \in B^{c_\mu}$ i.e., $f(x) \in B^{c_\mu}$. So $f(c_\mu(A)) \subseteq B^{c_\mu}$. Thus $f(c_\mu(A)) \subseteq c_{\mu_{\mathfrak{X}}}(f(A))$. Hence $f(c_\mu(A)) \subseteq (c_{\mu_{\mathfrak{X}}}(f(A))) \bigcap f(X)$.

For the reverse assertion, take $\mathcal{H} \in (c_{\mu_{\mathfrak{X}}}(f(A))) \bigcap f(X)$. Now as $\mathcal{H} \in f(X)$, $\mathcal{H} = \mathcal{H}_{c_\mu}(x)$, for some $x \in X$. Now $a \in c_\mu(A)$, $\forall\, a \in A \implies A \notin \mathcal{H}_{c_\mu}(a)$, $\forall\, a \in A \implies \mathcal{H}_{c_\mu}(a) \in A^{c_\mu}$, $\forall\, a \in A$ i.e., $f(a) \in A^{c_\mu}$, $\forall\, a \in A$. So $f(A) \subseteq A^{c_\mu}$. Now as $\mathcal{H} = \mathcal{H}_{c_\mu}(x) \in c_{\mu_{\mathfrak{X}}}(f(A))$, $\mathcal{H}_{c_\mu}(x) \in A^{c_\mu} \implies A \notin \mathcal{H}_{c_\mu}(x) \implies x \in c_\mu(A) \implies f(x) \in f(c_\mu(A))$ i.e., $\mathcal{H}_{c_\mu}(x) \in f(c_\mu(A))$ i.e., $\mathcal{H} \in f(c_\mu(A))$. Thus, $(c_{\mu_{\mathfrak{X}}}(f(A)) \bigcap f(X)) \subseteq f(c_\mu(A))$. So $f(c_\mu(A)) = c_{\mu_{\mathfrak{X}}}(f(A)) \bigcap f(X)$.

Lastly we claim that, $c_{\mu_{\mathfrak{X}}}(f(X)) = \mathfrak{X}$. Indeed, for $B \subseteq X$ with $f(X) \subseteq B^{c_\mu} \implies B \notin \mathcal{H}_{c_\mu}(x)$, $\forall x \in X \implies x \in c_\mu(B)$, $\forall x \in X \implies X \subseteq c_\mu(B)$ i.e., $X = c_\mu(B)$. Now, $X \notin \mathcal{H}$, $\forall\, \mathcal{H} \in \mathfrak{X}$. So, $c_\mu(B) \notin \mathcal{H}$, $\forall\, \mathcal{H} \in \mathfrak{X}$. In turn, $B \notin \mathcal{H}$, $\forall\, \mathcal{H} \in \mathfrak{X}$, as $\mathcal{H}$ is closure preserving $\forall\, \mathcal{H} \in \mathfrak{X}$. Hence $\mathcal{H} \in B^{c_\mu}$, $\forall\, \mathcal{H} \in \mathfrak{X}$. Thus $\mathcal{H} \in c_{\mu_{\mathfrak{X}}}(f(X))$, $\forall\, \mathcal{H} \in \mathfrak{X}$ i.e., $X_{\mathfrak{X}} \subseteq c_{\mu_{\mathfrak{X}}}(f(X))$. So, $c_{\mu_{\mathfrak{X}}}(f(X)) = \mathfrak{X}$. Hence, $(\mathfrak{X},\mu_{\mathfrak{X}})$ forms an extension of (X,μ), via the function f.

Result 4.9. (i) $\phi^{c_\mu} = \phi$.

(ii) $B \subseteq A \implies B^{c_\mu} \subseteq A^{c_\mu}$.

(iii) $c_{\mu_{\mathfrak{X}}}(f(A)) = A^{c_\mu}$.

Proof.

(i) As every hereditary class contains ϕ, the result follows immediately.

(ii) Take $\mathcal{H} \in B^{c_\mu}$. So $B \notin \mathcal{H}$. Therefore $\mathcal{H}$ being a hereditary class, $A \notin \mathcal{H}$. Hence $\mathcal{H} \in A^{c_\mu}$ and thus the result follows.

(iii) Take $A \subseteq X$. Then from the proof of the preceding theorem it is clear that $c_{\mu_{\mathfrak{X}}}(f(A)) \subseteq A^{c_\mu}$. Now take $\mathcal{H} \in A^{c_\mu}$ and $B \subseteq X$ such that $f(A) \subseteq B^{c_\mu}$. Then again from the proof of the preceding theorem it can be shown that $B \notin \mathcal{H}$. So $\mathcal{H} \in B^{c_\mu}$. Thus, $A^{c_\mu} \subseteq B^{c_\mu}$. So, $A^{c_\mu} \subseteq c_{\mu_{\mathcal{H}}}(f(A)$. Hence we are through.

Theorem 4.10. Let (X,μ) be a μ-T_0 GTS and $(\mathfrak{X},\mu_{\mathfrak{X}})$ be the extension of (X,μ) via the function f, as constructed in Theorem 4.8 above. Then we have the following:

(i) If $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{X}$ such that $\mathcal{H}_1 \neq \mathcal{H}_2$ then $S(\mathcal{H}_1) \neq S(\mathcal{H}_2)$.

(ii) The collection $\{c_{\mu_{\mathfrak{X}}}(f(A)) : A \subseteq X\}$ forms a base for the generalized closed sets of $(\mathfrak{X},\mu_{\mathfrak{X}})$.

Proof.

(i) $S(\mathcal{H}_1) = \{A \subseteq X : \mathcal{H}_1 \notin c_{\mu_{\mathfrak{X}}}(f(A))\} = \{A \subseteq X : \mathcal{H}_1 \notin A^{c_\mu}\} = \{A \subseteq X : A \in \mathcal{H}_1\} = \mathcal{H}_1$, Similarly, $S(\mathcal{H}_2) = \mathcal{H}_2$. So, $S(\mathcal{H}_1) \neq S(\mathcal{H}_2)$, as $\mathcal{H}_1 \neq \mathcal{H}_2$.

(ii) From Result 4.9 it follows that $c_{\mu_{\mathfrak{X}}}(f(A)) = A^{c_\mu}$. Again from Theorem 4.8, we know that the collection $\{A^{c_\mu} : A \subseteq X\}$ forms a base for the generalized closed sets of $(\mathfrak{X},\mu_{\mathfrak{X}})$. Hence the result follows.

Now we define a certain class of extensions by virtue of the two properties, noted in the above theorem.

Definition 4.11. Let, (Y, μ') be an extension of a *GTS* (X, μ) via the function f. Then this extension is called a primary extension if the following hold:

(i) For $y_1, y_2 \in Y$ with $y_1 \neq y_2$, $S(y_1) \neq S(y_2)$.

(ii) The collection $\{c_{\mu_Y}(f(A)) : A \subseteq X\}$ forms a base for the generalized closed sets of (Y, μ_Y).

Now from Theorem 4.10, it is clear that the extension constructed in Theorem 4.8 is a primary extension. At this point the purpose of the above definition might be questioned quite logically. The next couple of results would justify the purpose.

Theorem 4.12. Let, (X, μ) be a μ-T_0 *GTS*. Then any two primary extensions of (X, μ) with identical strength systems are equivalent.

Proof. Let, (Y, μ_Y) and (Z, μ_Z) be two primary extension of (X, μ) via the functions f_Y and f_Z respectively. Now since $S(Y) = S(Z)$, for each $y \in Y$, we can choose $z \in Z$ such that $S(y) = S(z)$. Let us define $f : Y \to Z$ as, $f(y) = z$. Now as both the extensions are primary, f is well defined, as well as one-to-one and onto. Take $A \subseteq X$ and $y \in c_{\mu_Y}(f_Y(A))$. So, $A \notin S(y) = S(f(y)) \implies f(y) \in c_{\mu_Z}(f_Z(A))$. Thus, $f(c_{\mu_Y}(f_Y(A))) \subseteq c_{\mu_Z}(f_Z(A))$. Similarly as f is bijective, it can also be shown that $f^{-1}(c_{\mu_Z}(f_Z(A))) \subseteq c_{\mu_Y}(f_Y(A))$ i.e., $c_{\mu_Z}(f_Z(A)) \subseteq f(c_{\mu_Y}(f_Y(A)))$. Hence, $f(c_{\mu_Y}(f_Y(A))) = c_{\mu_Z}(f_Z(A))$. Now the two collections $\{c_{\mu_Y}(f_Y(A))\}$ and $\{c_{\mu_Z}(f_Z(A))\}$ form bases for (Y, μ_Y) and (Z, μ_Z) respectively. So, the bijective function f corresponds these two bases in one-one manner. Thus, f turns out to be a g-homeomorphism and hence the result follows.

Theorem 4.13. Any primary extension of a μ-T_0 *GTS* (X, μ) is equivalent to an extension of the same type as the one in Theorem 4.8.

Proof. Let (Y, μ') be a primary extension of the *GTS* (X, μ) via the function h. Then from Remark 2.27 it follows that $S(y)$ is a closure preserving hereditary class on X and also $X \notin S(y)$, for each $y \in Y$. Again, from Theorem 4.5 it follows that for each $x \in X$, $S(h(x)) = \mathcal{H}_{c_\mu}(x)$. Next consider the collection $\mathfrak{X} = \{S(y) : y \in Y\}$ and construct a *GT*, say $\mu_{\mathfrak{X}}$, on this collection as constructed in Theorem 4.8. Now both (Y, μ') and $(\mathfrak{X}, \mu_{\mathfrak{X}})$ are primary extensions of (X, μ), via the respective functions. Again from the proof of Theorem 4.10(i), it is clear that $S(S(y)) = S(y)$. Thus both these extensions have the same strength system and hence from the previous theorem it follows that these two extensions are equivalent.

5. Compact Extension of *GTS*

In this section we confine ourselves to a particular type of extension of a given *GTS*. Here we study those extensions which are compact with respect to the concerned generalized topology. For this we first consider the the following definitions.

Definition 5.1. [18] A *GTS* (X, μ) is said to be μ-compact if every μ-open cover of X has a finite subcover for X.

Now we consider certain characterizations of μ-compactness. For this first we recall the next two definitions from the literature.

Definition 5.2. [10] For a non-empty set X, a subcollection $\mathcal{F}$ of $\exp(X)$ is said to satisfy finite intersection property if for any finite subcollection $\{F_i : i = 1, 2, ..., n\}$ of $\mathcal{F}$, $\bigcap_{i=1}^{n} F_i \neq \phi$.

Definition 5.3. [10] For a non-empty set X, a non-void subfamily $\mathcal{U}$ of $\exp(X)$ is said to form an ultrafilter on X if

(i) $A \in \mathcal{U}$ and $A \subseteq B \implies B \in \mathcal{U}$.

(ii) $A, B \in \mathcal{U} \implies (A \cap B) \in \mathcal{U}$

and

(iii) for each subset A of X, either $A \in \mathcal{U}$ or $(X \setminus A) \in \mathcal{U}$.

Then we have the the following characterizations of μ-compactness.

Theorem 5.4. [19] A *GTS* (X, μ) is μ-compact iff every collection of μ-closed subsets of X with finite intersection property has a non-empty intersection.

Theorem 5.5. A *GTS* (X, μ) is μ-compact iff every ultrafilter on it converges.

Proof. The proof is similar to the corresponding result of compactness.

Definition 5.6. A hereditary class (or an ideal) $\mathcal{H}$ on a *GTS* (X, μ) is said to be complement conjoint if for $A_i \in (\exp(X) \setminus \mathcal{H})$ $(i = 1, 2, ...n)$, $n \in \mathbb{N}$, $\bigcap_{i=1}^{n} c_\mu(A_i) \neq \phi$.

Theorem 5.7. A *GTS* (X, μ) is μ-compact iff for every complement conjoint hereditary class (or ideal) $\mathcal{H}$ on X, $\exists\, x \in X$ such that $\mathcal{H}_{c_\mu}(x) \subseteq \mathcal{H}$.

Proof. Let (X, μ) be a μ-compact *GTS* and $\mathcal{H}$ a complement conjoint hereditary class on X. Then consider the collection $\{c_\mu(A) : A \in (\exp(X) \setminus \mathcal{H})\}$. Now, as $\mathcal{H}$ is complement conjoint, this collection of c_μ-closed subsets of X has finite intersection property. So as X is μ-compact, $\exists\, x \in X$ such that $x \in \bigcap\{c_\mu(A) : A \in (exp(X) \setminus \mathcal{H})\}$. Thus, $A \notin \mathcal{H}_{c_\mu}(x)$, $\forall\, A \in (\exp(X) \setminus \mathcal{H})$. Hence $\mathcal{H}_{c_\mu}(x) \subseteq \mathcal{H}$.

Conversely, suppose that the condition holds and let $\mathcal{U}$ be an ultrafilter on X. Now consider the collection $\mathcal{H} = \{A \subseteq X : (X \setminus A) \in \mathcal{U}\}$. Then it can easily be shown that $\mathcal{H}$ is an ideal and hence a hereditary class on X. Now take $A_i \in (\exp(X) \setminus \mathcal{H})$ $(i = 1, 2, ...n)$, $n \in \mathbb{N}$. Then $(X \setminus A_i) \notin \mathcal{U}, \forall i \in \{1, 2, ..., n\}$. Then $\mathcal{U}$ being an ultrafilter, $\bigcup_{i=1}^{n}(X \setminus A_i) \notin \mathcal{U}$, i.e., $X \setminus \bigcap_{i=1}^{n} A_i \notin \mathcal{U}$. So, $\bigcap_{i=1}^{n} A_i \notin \mathcal{H} \implies \bigcap_{i=1}^{n} A_i \neq \phi$ and hence $\bigcap_{i=1}^{n} c_\mu(A_i) \neq \phi$. Thus the hereditary class $\mathcal{H}$ is complement conjoint. Then by the given condition, $\exists\, x \in X$ such that $\mathcal{H}_{c_\mu}(x) \subseteq \mathcal{H}$. Now consider an arbitrary $G \in \mu$ such that $x \in G$. Then clearly, $x \notin c_\mu(X \setminus G) \implies (X \setminus G) \in \mathcal{H}_{c_\mu}(x) \subseteq \mathcal{H} \implies G \in \mathcal{U}$ and hence $\mathcal{U}$ converges to x. Thus X is μ-compact.

Definition 5.8. A collection $\mathfrak{X}$ of hereditary classes on a non-empty set X is said to satisfy 'Condition $*$' if whenever $A_i \in (\exp(X) \setminus \mathcal{H})$ $(i = 1, 2, ...n)$, $n \in \mathbb{N}$, for some hereditary class $\mathcal{H}$ on X implies $\exists\, \mathcal{H}' \in \mathfrak{X}$ such that $A_i \notin \mathcal{H}'$, $\forall\, i \in \{1, 2, ...n\}$, then $\exists\, \mathcal{H}_0 \in \mathfrak{X}$ such that $\mathcal{H}_0 \subseteq \mathcal{H}$.

Theorem 5.9. Any primary extension $(\mathfrak{X}, \mu_{\mathfrak{X}})$ of a μ-T_0 *GTS* (X, μ) is $\mu_{\mathfrak{X}}$-compact iff $\mathfrak{X}$ satisfies 'Condition $*$'.

Proof. By virtue of Theorem 4.13, any primary extension of (X, μ) can be considered to be of the form $(\mathfrak{X}, \mu_{\mathfrak{X}})$, where $\mathfrak{X}$ is a collection of closure preserving hereditary classes on X, containing all the adherence free hereditary classes and also no member of $\mathfrak{X}$ contains X. Also consider f to be the corresponding extension function. Firstly let $(\mathfrak{X}, \mu_{\mathfrak{X}})$ be $\mu_{\mathfrak{X}}$-compact. Now consider a hereditary class $\mathcal{H}$ on X, satisfying $A_i \in (\exp(X) \setminus \mathcal{H})$ $(i = 1, 2, ...n)$, $n \in \mathbb{N} \implies \exists\, \mathcal{H}' \in \mathfrak{X}$ such that $A_i \notin \mathcal{H}'$, $\forall\, i \in \{1, 2, ...n\}$. We have to find some $\mathcal{H}_0 \in \mathfrak{X}$ such that $\mathcal{H}_0 \subseteq \mathcal{H}$. To do so first of all consider the collection $\mathfrak{A} = \{\alpha \subseteq \mathfrak{X} : f(A) \cap (\mathfrak{X} \setminus \alpha) \neq \phi, \forall\, A \notin \mathcal{H}\}$. Now let $\alpha, \beta \in \mathfrak{X}$ such that $\beta \subseteq \alpha$ and $\alpha \in \mathfrak{A}$. Take $A \subseteq X$ such that $A \notin \mathcal{H}$. Then $f(A) \cap (\mathfrak{X} \setminus \alpha) \neq \phi$, as $\alpha \in \mathfrak{A}$. Again, $\beta \subseteq \alpha \implies (f(A) \cap (\mathfrak{X} \setminus \alpha)) \subseteq (f(A) \cap (\mathfrak{X} \setminus \beta))$. So, $f(A) \cap (\mathfrak{X} \setminus \beta) \neq \phi$. Thus, $\beta \in \mathfrak{A}$ and hence $\mathfrak{A}$ is a hereditary class on $\mathfrak{X}$. Also it is quite obvious that $\mathfrak{X} \notin \mathfrak{A}$. Next let $\alpha_i \in (\exp(\mathfrak{X}) \setminus \mathfrak{A})$, for $i \in \{1, 2, ...n\}$, $n \in \mathbb{N}$. Then for each $i \in \{1, 2, ..., n\}$, $\exists\, A_i \subseteq X$ such that $A_i \notin \mathcal{H}$ and $f(A_i) \cap (\mathfrak{X} \setminus \alpha_i) = \phi$ i.e., $f(A_i) \subseteq \alpha_i$. Now, ${A_i}^{c_\mu} = c_{\mu_{\mathfrak{X}}}(f(A_i))$, $\forall\, i \in \{1, 2, ..., n\}$. So, $\bigcap_{i=1}^{n} {A_i}^{c_\mu} = \bigcap_{i=1}^{n} c_{\mu_{\mathfrak{X}}}(f(A_i)) \subseteq \bigcap_{i=1}^{n} c_{\mu_{\mathfrak{X}}}(\alpha_i)$. Again, $A_i \notin \mathcal{H}$, $\forall\, i \in \{1, 2, ..., n\}$. So, $\mathcal{H} \in {A_i}^{c_\mu}$, $\forall\, i \in \{1, 2, ..., n\}$ i.e., $\mathcal{H} \in \bigcap_{i=1}^{n} {A_i}^{c_\mu}$. Hence, $\bigcap_{i=1}^{n} {A_i}^{c_\mu} \neq \phi$ and thus $\bigcap_{i=1}^{n} c_{\mu_{\mathfrak{X}}}(\alpha_i) \neq \phi$. So, $\mathfrak{A}$ is complement conjoint. Now $(\mathfrak{X}, \mu_{\mathfrak{X}})$ being $\mu_{\mathfrak{X}}$-compact, $\exists\, \mathcal{H}_0 \in \mathfrak{X}$ such that $\mathcal{H}_{c_{\mu_{\mathfrak{X}}}}(\mathcal{H}_0) \subseteq \mathfrak{A}$. Now $A \notin \mathcal{H} \implies f(A) \notin \mathfrak{A} \implies f(A) \notin \mathcal{H}_{c_{\mu_{\mathfrak{X}}}}(\mathcal{H}_0) \implies \mathcal{H}_0 \in c_{\mu_{\mathfrak{X}}}(f(A)) = A^{c_\mu} \implies A \notin \mathcal{H}_0$ Thus, $\mathcal{H}_0 \subseteq \mathcal{H}$. Thus $\mathfrak{X}$ satisfies 'Condition $*$'.

Conversely, suppose that the given condition holds and let $\mathfrak{A}$ be a complement conjoint ideal on $\mathfrak{X}$. In view of Theorem 5.7 we have to find out some $\mathcal{H}_0 \in \mathfrak{X}$ such that $\mathcal{H}_{c_{\mu_{\mathfrak{X}}}}(\mathcal{H}_0) \subseteq \mathfrak{A}$. Take $\mathfrak{A}_1 = \{A \subseteq X : f(A) \in \mathfrak{A}\}$

and $\mathfrak{A}_2 = \{A \subseteq X : A^{c_\mu} \setminus f(X) \in \mathfrak{A}\}$. Now, $\phi \in \mathfrak{A}_1$ and $\phi \in \mathfrak{A}_2$. So, $\mathfrak{A}_1 \neq \phi \neq \mathfrak{A}_2$. Then it can easily be shown that both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ form ideals (and hence hereditary classes) on X. So, $\mathcal{H} = \mathfrak{A}_1 \cap \mathfrak{A}_2$ also forms an ideal (and hence a hereditary class) on X. Next, let $A_i \in (\exp(X) \setminus \mathcal{H})$ $(i = 1, 2, ...n)$, $n \in \mathbb{N}$. Then $f(A_i) \notin \mathfrak{A}$ or $(A_i{}^{c_\mu} \setminus f(X)) \notin \mathfrak{A}$. Without loss of any generality we may assume that $f(A_i) \notin \mathfrak{A}$, for $i = 1, 2, ...m$ and $(A_i{}^{c_\mu} \setminus f(X)) \notin \mathfrak{A}$, for $i = m+1, m+2, ...n$, $m(\leq n) \in \mathbb{N}$. Now $\mathfrak{A}$ being complement conjoint, $(\bigcap\limits_{i=1}^{m} c_{\mu_\mathfrak{X}}(f(A_i))) \bigcap (\bigcap\limits_{i=m+1}^{n} c_{\mu_\mathfrak{X}}(A_i{}^{c_\mu} \setminus f(X))) \neq \phi \implies (\bigcap\limits_{i=1}^{m} c_{\mu_\mathfrak{X}}(f(A_i))) \bigcap (\bigcap\limits_{i=m+1}^{n} c_{\mu_\mathfrak{X}}(A_i{}^{c_\mu})) \neq \phi$, as $(A_i{}^{c_\mu} \setminus f(X)) \subseteq A_i{}^{c_\mu}$. So, $(\bigcap\limits_{i=1}^{m} A_i{}^{c_\mu}) \bigcap (\bigcap\limits_{i=m+1}^{n} A_i{}^{c_\mu}) \neq \phi$, since $c_{\mu_\mathfrak{X}}(f(A_i)) = A_i{}^{c_\mu}$, $\forall\ i = 1, 2, ...n$ and $c_\mu(A) = c_\mu(c_\mu(A))$, $\forall\ A \subseteq X$. Thus, $\bigcap\limits_{i=1}^{n} A_i{}^{c_\mu} \neq \phi$. Hence $\exists\ \mathcal{H}' \in \mathfrak{X}$ such that $\mathcal{H}' \in \bigcap\limits_{i=1}^{n} A_i{}^{c_\mu}$. So, $A_i \notin \mathcal{H}'$, $\forall\ i \in \{1, 2, ...n\}$. Now as $\mathfrak{X}$ satisfies 'Condition $*$', $\exists\ \mathcal{H}_0 \in \mathfrak{X}$ such that $\mathcal{H}_0 \subseteq \mathcal{H}$. Next we have to show that $\mathcal{H}_{c_{\mu_\mathfrak{X}}}(\mathcal{H}_0) \subseteq \mathfrak{A}$. To do so let $\alpha \subseteq \mathfrak{X}$ such that $\alpha \notin \mathfrak{A}$. Now $\alpha = (\alpha \cap f(X)) \bigcup (\alpha \setminus f(X))$. So, either $(\alpha \cap f(X)) \notin \mathfrak{A}$ or $(\alpha \setminus f(X)) \notin \mathfrak{A}$, as $\mathfrak{A}$ is an ideal. If $(\alpha \cap f(X)) \notin \mathfrak{A}$ then $f^{-1}(\alpha) \notin \mathcal{H}$, as $f(f^{-1}(\alpha)) = \alpha \cap f(X)$. So $f^{-1}(\alpha) \notin \mathcal{H}_0 \implies \mathcal{H}_0 \in (f^{-1}(\alpha))^{c_\mu} = c_{\mu_\mathfrak{X}}(f(f^{-1}(\alpha))) = c_{\mu_\mathfrak{X}}(\alpha \cap f(X)) \subseteq c_{\mu_\mathfrak{X}}(\alpha) \implies \alpha \notin \mathcal{H}_{c_{\mu_\mathfrak{X}}}(\mathcal{H}_0)$. For the remaining case $(\alpha \setminus f(X)) \notin \mathfrak{A}$. If so then take $A \subseteq X$ such that $\alpha \subseteq A^{c_\mu}$. So, $(\alpha \setminus f(X)) \subseteq (A^{c_\mu} \setminus f(X))$. Hence $(A^{c_\mu} \setminus f(X)) \notin \mathfrak{A}$, as $\mathfrak{A}$ is an ideal. Hence $A \notin \mathfrak{A}_2 \Longrightarrow A \notin \mathcal{H}$, since $\mathcal{H} = \mathfrak{A}_1 \cap \mathfrak{A}_2$ So, $A \in \mathcal{H}_0$, as $\mathcal{H}_0 \subseteq \mathcal{H}$. Thus $\mathcal{H}_0 \in A^{c_\mu}$. So $\mathcal{H}_0 \in \bigcap A^{c_\mu}$ for $A \subseteq X$ with $\alpha \subseteq A^{c_\mu}$. So, $\mathcal{H}_0 \in c_{\mu_\mathfrak{X}}(\alpha) \implies \alpha \notin \mathcal{H}_{c_{\mu_\mathfrak{X}}}(\mathcal{H}_0)$. Thus in any case, $\alpha \notin \mathcal{H}_{c_{\mu_\mathfrak{X}}}(\mathcal{H}_0)$. So $\mathcal{H}_{c_{\mu_\mathfrak{X}}}(\mathcal{H}_0) \subseteq \mathfrak{A}$ and hence $(\mathfrak{X}, \mu_\mathfrak{X})$ is compact.

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