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Common fixed point theorems in metric space using new CLR property

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Abstract

In this paper we present a new class of functions called weakly $CLR^{*}_{(f,g),T}$ property which generalizes the class of mappings already defined in the literatures [8]. We obtain a result on Common fixed point theorem in metric space for mappings satisfying weakly $CLR^{*}_{(f,g),T}$ property. Some examples to justify our results are given.

Keywords: Weakly $CLR_{(f,g),T}$ property, common CLR property, $CLR_{(f,g),T}$ property, weakly Compatible mappings, implicit function, coincident point, fixed point.
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1. Introduction Preliminaries

In 1986, Jungck[3] gave the concept of compatible mappings.

Definition 1.1. A pair of self-mapping $f, h \colon X \to X$ is compatible if $\lim_{n\to\infty} d(fhx_n, hfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X, such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z$, for some $z \in X$.

In 1996, the same author [4] generalized this concept to weakly compatilble maps to study common fixed point theorems.

Definition 1.2. A pair of self-mapping $f, h : X \to X$ is weakly compatible if they commute at their coincidence points, that is, if there exists a point $x \in X$, such that $fhx = hfx$, whenever $fx = hx$.

In the study of common fixed points of weakly compatible mappings, one often requires the assumptions like completeness of the space or subspace or the continuity of mappings, besides some contractive conditions. Aamri and El Moutawakil [5] introduced the notion of (E.A) property, which requires only the closeness of the subspace.

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Definition 1.3. [5] Let (X, d) be a metric space and $f, h : X \to X$ be two self-maps. The pair (f, h) is said to satisfy the (E.A) property if there exists a sequence $\{x_n\}$ in X and some $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z \in X.$$

Liu et al. [10] extended the (E.A) property to the common (E.A) property as follows.

Definition 1.4. Let (X, d) be a metric space and $f, g, h, J : X \to X$ be four self-maps. The pairs (f, h) and (g, J) satisfy the common (E.A) property if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X and some $t \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = t \in X.$$

Definition 1.5. [2] Let (X, d) be a metric space and $f, h \colon X \to X$ be self-maps. The pair (f, h) is called subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z, \ \ for\ some\ z \in X.$$

and

$$\lim_{n\to\infty} fhx_n = fz, \ \lim_{n\to\infty} hfx_n = hz.$$

It follows that if the pair of mapping (f, h) is compatible and subseqentially continuous then the existence of coincident point is obvious. Since from compatibility and subsequential continuity we have,

$$\lim_{n\to\infty} d(fhx_n, hfx_n) = 0.$$

and from subsequential continuity we have, $d(hz, fz) = 0$.
Sintunavarat and Kumam [9] introduced following property, which never requires any condition on closeness of the space or subspace.

Definition 1.6. Let (X, d) be a metric space and $f, h \colon X \to X$ be self-maps. The pair (f, h) said to satisfy the common limit in the range of h property if

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = h(x), \ \ for\ some\ x \in X.$$

Here we can observe that if the pair of mapping (f, h) satisfy (E.A) property together with the condition that $h(X)$ is closed then the pair also satisfies the common limit in the range of h property.
Imdad et al. [6] introduced the common (CLR) property which is an extension of the (CLR) property.

Definition 1.7. Let (X, d) be a metric space and $f, g, h, J : X \to X$ be four self-maps. The pairs (f, h) and (g, J) satisfy the common limit range property with respect to mappings h and J, denoted by $CLR_{(h,J)}$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = t,$$

where $t \in h(X) \cap J(X)$.

Valeriu Popa, [8] introduced a new class of mappings as follows and proved some results for common fixed point theorem.

Definition 1.8. Let (X, d) be a metric space and $f, g, T : X \to X$ be three self-maps. The pair (f, g) is said to satisfy common limit range property with respect to T, denoted $CLR_{(f,g),T}$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \ \ for\ some\ t \in g(X) \cap T(X).$$

The author have given a remark that if (f, h) and (g, T) satisfy the common limit range property with respect to mappings h and T, then (f, h) satisfy the $CLR_{(f,h),T}$. But the converse is not true.

Definition 1.9. [1] Let $\mathcal{F}$ be the family of lower semi - continuous functions $F(t1, ..., t6)$: $R_+^6 \to R$ satisfying the following conditions:

$(F1) : F(t, 0, t, 0, 0, t) > 0, \ \forall t > 0;$
$(F2) : F(t, 0, 0, t, t, 0) > 0, \ \forall t > 0;$
$(F3) : F(t, t, 0, 0, t, t) > 0, \ \forall t > 0.$

For examples we refer [1] to the readers.

Definition 1.10. Let f and g be self-maps on set X. If $fx = gx = w$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 1.11. [7] An altering distance is a function ψ: $[0, 1) \to [0, 1)$ satisfying:

1) ψ is continuous and increasing ;
2) $\psi(t) = 0$ if and only if $t = 0$.

$C(f, g) = \{x : fx = gx\}$ is the collection of all coincidence points of selfmaps f and g of a metric space X.

Theorem 1.12. [8] Let (X, d) be a metric space and f, g, h and T be self mappings of X satisfying the inequality

$$F(\psi(d(fx, hy)), \psi(d(gx, Ty)), \psi(d(gx, fx)), \psi(d(Ty, hy)), \psi(d(gx, hy)), \psi(d(Ty, fx))) \le 0,$$

for all $x, y \in X$, $F \in \mathcal{F}$ and ψ is an altering distance.

If $f, g,$ and T satisfy $CLR_{(f,g),T}$ - property, then $C(f, g) \neq \phi,$ *and* $C(h, T) \neq \phi$. Moreover, if (f, g) and (h, T) are weakly compatible, then f, g, h and T have a unique common fixed point.

Let us consider the following notations.

$$m_1(x, y) = \{d(Tgx, Ty), d(Tfx, Tgx), d(hy, Ty), \frac{d(Tgx,hy)+d(Ty,Tfx)}{2}, \frac{d(Tfx,Tgx)d(hy,Ty)}{1+d(Tgx,Ty)}, \frac{d(Tfx,Ty)d(hy,Tgx)}{1+d(Tgx,Ty)},$$
$$d(Tfx, Tgx)\frac{1+d(Tgx,hy)+d(Ty,Tfx)}{1+d(Tfx,Tgx)+d(Ty,hy)}\}.$$

$$m_2(x, y) = \{d(Tgx, Ty), d(Tfx, Tgx), d(hy, Ty), \frac{d(Tgx,hy)+d(Ty,Tfx)}{2}, \frac{1+d(Tfx,Tgx)}{1+d(Tgx,Ty)}d(Ty, hy), \frac{1+d(hy,Ty)}{1+d(Tgx,Ty)}d(Tgx, Tfx),$$
$$d(Ty, hy)\frac{1+d(Tgx,hy)+d(Ty,Tfx)}{1+d(Tfx,Tgx)+d(Ty,hy)}\}.$$

$$m_3(x, y) = \{d(Tgx, Ty), d(Tfx, Tgx), d(hy, Ty), \frac{d(Tgx,hy)+d(Ty,Tfx)}{2}, \frac{d(Tfx,Tgx)d(hy,Ty)}{1+d(Tfx,hy)}, \frac{d(Tfx,Ty)d(hy,Tgx)}{1+d(hy,Tfx)},$$
$$d(Tgx, Tfx)\frac{1+d(Tgx,hy)+d(Ty,Tfx)}{1+d(Tfx,Tgx)+d(Ty,hy)}\}.$$

$$m_4(x, y) = \{d(Tgx, Ty), d(Tfx, Tgx), d(hy, Ty), \frac{d(Tgx,hy)+d(Ty,Tfx)}{2}, \frac{d(Tfx,Tgx)d(hy,Ty)}{1+d(Tgx,Ty)}, \frac{d(Tfx,Ty)d(hy,Tgx)}{1+d(Tgx,Ty)},$$
$$\frac{d(Tfx,Ty)d(hy,Tgx)}{1+d(Tfx,hy)}\}.$$

Let us consider some classes of mappings.

$\Phi_1 = \{\varphi$: $R^+ \to R^+$ satisfies that φ is Lebesgue integrable, summable on each compact subset of R^+ and for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0\}$.
$\Phi_2 = \{\varphi$: $R^+ \to R^+$ is nondecreasing continuous function on $R^+ - 0$ and $\varphi(t) = 0 \iff t = 0$ }.
$\Phi_3 = \{\varphi$: $R^+ \to R^+$ is lower semicontinuous function and $\varphi(t) > 0$ for each $t > 0\} = \{\varphi$: $R^+ \to R^+$ is lower semicontinuous function and $\varphi(t) = 0$ iff $t = 0\}$.

2. Common Fixed Point for weakly $CLR^*_{(f,g),T}$

Here in this paper we introduce a new class of mapping called weakly $CLR^*_{(f,g),T}$ property defined as follows:

Definition 2.1. Let (X, d) be a metric space and $f, g, T : X \to X$ be three self-maps. The pair (f, g) is said to satisfy weakly common limit range property with respect to T, denoted $CLR^*_{(f,g),T}$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tgx_n = t, \ \ for \ \ some \ t \in g(X) \cap T(X).$$

If in the above definition we take $T(x) = x$, then we get the $CLR_{(f,g),T}$ property defined in [8].

We present few examples that will illustrate that weakly $CLR^*_{(f,g),T}$ is more general than $CLR_{(f,g),T}$.

So if the mappings f, g, T satisfies $CLR_{(f,g),T}$, then it satisfies weakly $CLR^*_{(f,g),T}$. But the converse is not true. Consider the following examples. In example 2.2 we show that the mappings satisfy both $CLR_{(f,g),T}$ and weakly $CLR^*_{(f,g),T}$ and example 2.3 shows that the mappings do not satisfy $CLR_{(f,g),T}$ but they satisfy weakly $CLR^*_{(f,g),T}$.

Example 2.2. Let $X = [0, 2)$ be a metric space with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$ and f, g and $T\colon X \to X$, defined by

$$f(x) = \begin{cases} 1-x, & \text{for } x \in [0,1), \\ \frac{3}{2}, & \text{for } x \in [1,2), \end{cases} \quad g(x) = \begin{cases} 1-x, & \text{for } x \in [0,1), \\ \frac{3}{4}, & \text{for } x \in [1,2), \end{cases} \quad T(x) = \begin{cases} x+\frac{1}{2}, & \text{for } x \in [0,1), \\ 1, & \text{for } x \in [1,2), \end{cases}$$

Let $\{x_n\} = \{\frac{1}{n}\}$ be a sequence in X. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} f(\{\frac{1}{n}\}) = \lim_{n\to\infty} \{1 - \frac{1}{n}\} = 1;$$

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} g(\{\frac{1}{n}\}) = \lim_{n\to\infty} \{1 - \frac{1}{n}\} = 1;$$

Thus $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1$. Also $1 \in g(X) \cap T(X)$. That is, the pair (f, g) satisfy $CLR_{(f,g),T}$.

Let $\{x_n\} = \{\frac{1}{2} + \frac{1}{n}\}$ be a sequence in X. Then,

$$\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tf(\{\frac{1}{2} + \frac{1}{n}\}) = \lim_{n\to\infty} T(\frac{1}{2} - \frac{1}{n}) = \lim_{n\to\infty} (\frac{1}{2} - \frac{1}{n} + \frac{1}{2}) = 1;$$

$$\lim_{n\to\infty} Tgx_n = \lim_{n\to\infty} Tg(\{\frac{1}{2} + \frac{1}{n}\}) = \lim_{n\to\infty} T(\frac{1}{2} - \frac{1}{n}) = \lim_{n\to\infty} (\frac{1}{2} - \frac{1}{n} + \frac{1}{2}) = 1;$$

$$\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tgx_n = 1.$$

Now $1 \in g(X) \cap T(X)$. That is the pair (f, g) satisfy weakly $CLR^*_{(f,g),T}$.

Example 2.3. Let $X = R^+ \cup \{0\}$, be a set and f, g and $T\colon X \to X$, defined by

$$f(x) = \begin{cases} 1, & \text{for } x = 1, \\ x+1, & \text{for } x \neq 1, \end{cases} \quad g(x) = \begin{cases} 1, & \text{for } x = 1, \\ 2x, & \text{for } x \neq 1, \end{cases} \quad T(x) = \begin{cases} 0, & \text{for } x = 1, \\ x+2, & \text{for } x \neq 1. \end{cases}$$

Let $\{x_n\} = \{1\}, \forall n \in N$ be a sequence in X. Then

$$\lim_{n\to\infty} fx_n = 1, \ \lim_{n\to\infty} gx_n = 1;$$

Thus, here

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1.$$

But $1 \notin T(X) = \{0\} \cup [2, \infty)$. That is, the pair (f, g) do not satisfy $CLR_{(f,g),T}$,

$$\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tf(1) = \lim_{n\to\infty} T(1) = 0;$$
$$\lim_{n\to\infty} Tgx_n = \lim_{n\to\infty} Tg(1) = \lim_{n\to\infty} T(1) = 0;$$
$$\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tgx_n = 0.$$

Also, we have $0 \in g(X) \cap T(X)$. That is the pair $(f,\ g)$ satisfy weakly $CLR^*_{(f,g),T}$. If in addition we consider another mapping

$$h(x) = \begin{cases} 0, & \text{for } x = 1, \\ 2x + 2, & \text{for } x \neq 1, \end{cases}$$

Let $\{y_n\} = \{\frac{1}{n}\}, \forall n \in N$ be a sequence in X. Then

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} hy_n = 2 \neq 1 = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n;$$

Thus f, g, h and T do not satisfy the $CLR_{(g,T)}$.

Theorem 2.4. Let $(X,\ d)$ be a metric space and $f,\ g,\ h,\ T : X \to X$ such that (f, g) satisfy weakly $CLR^*_{(f,g),T}$ property and for each $x, y \in X$,

$$d(Tfx, hy) \leq a_1 d(Tgx, Ty) + a_2 d(Tfx, Tgx) + a_3 d(Ty, hy) + a_4 d(hy, Tgx) + a_5 d(Tfx, Ty).$$

where, $a_i \in (0, 1)$ for $i = 1, 2, 3, 4, 5$ and $\sum_{i=1}^{5} a_i < 1$. Then if T is one-one function and $(f,\ h)$ and $(g,\ T)$ is weakly compatible mappings then $f,\ g,\ h$ and T have a unique common fixed point in X.

Proof: Since $f,\ g,$ and $T\colon X \to X$ such that (f, g) satisfy weakly $CLR^*_{(f,g),T}$ property. There exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tgx_n = z,$$

for some $z \in g(X) \cap T(X)$. Let $z = T(u)$, for some $u \in X$. Taking $x = x_n$ and $y = u$ in the theorem we get,

$$d(Tfx_n, hu) \leq a_1\, d(Tgx_n, Tu) + a_2\, d(Tfx_n, Tgx_n) + a_3\, d(Tu, hu) + a_4\, d(hu, Tgx_n) + a_5\, d(Tfx_n, Tu).$$

Let $n \to \infty$, then we get,

$$\begin{aligned} d(z, hu) &\leq a_1\, d(z, Tu) + a_2\, d(z, z) + a_3\, d(Tu, hu) + a_4\, d(hu, z) + a_5\, d(z, Tu). \\ &\leq a_1\, d(z, z) + a_2\, d(z, z) + a_3\, d(z, hu) + a_4\, d(hu, z) + a_5\, d(z, z). \\ &\leq (a_3 + a_4)\, d(hu, z). \end{aligned}$$

Therefore, we get, $d(z, hu) = 0$. So $hu = z = Tu$.
So, $C(h, T) \neq \phi$. Suppose $hw = Tw$ such that $w \neq u$.
Then taking $x = x_n$ and $y = w$ in the theorem we get,

$$d(Tfx_n, hw) \leq a_1\, d(Tgx_n, Tw) + a_2\, d(Tfx_n, Tgx_n) + a_3\, d(Tw, hw) + a_4\, d(hw, Tgx_n) + a_5\, d(Tfx_n, Tw).$$

Let $n \to \infty$, then we get,

$$\begin{aligned} d(z, hw) &\leq a_1\, d(z, hw) + a_2\, d(z, z) + a_3\, d(hw, hw) + a_4\, d(hw, z) + a_5\, d(z, hw). \\ &\leq (a_1 + a_4 + a_5)\, d(z, hw). \end{aligned}$$

Therefore, we get, $d(z, hw) = 0$. So $hw = z = Tw$.
Which implies that the point of coincidence of (h, T) is unique. Since, (h, T) is weakly compatible mappings z is the unique common fixed point of (h, T).
So, we have,

$$hz = z = Tz.$$

Again $z \in g(X)$, let $g(v) = z$, for some $v \in X$.
Taking $x = v$ and $y = z$ in the Theorem we get,

$$\begin{aligned}
&\leq a_1\, d(Tgv, Tz) + a_2\, d(Tfv, Tgv) + a_3\, d(Tz, hz) + a_4\, d(hz, Tgv) + a_5\, d(Tfv, Tz).\\
&\leq a_1\, d(Tz, Tz) + a_2\, d(Tfv, Tz) + a_3\, d(Tz, Tz) + a_4\, d(z, Tz) + a_5\, d(Tfv, z).\\
&\leq (a_2 + a_5)\, d(Tfv, z).
\end{aligned}$$

Therefore, we get, $d(z, Tfv) = 0$. So $Tfv = z = Tz$.
Since T is one-one we get, $fv = z = gv$.
Therefore, $C(f, g) \neq \phi$. Suppose $fp = gp$, where $p \neq v$.
Taking $x = p$ and $y = z$ in the theorem we get,

$$\begin{aligned}
d(Tfp, hz) &\leq a_1\, d(Tgp, Tz) + a_2\, d(Tfp, Tgp) + a_3\, d(Tz, hz) + a_4\, d(hz, Tgp) + a_5\, d(Tfp, Tz).\\
&\leq a_1\, d(Tfp, z) + a_2\, d(Tfp, Tgp) + a_3\, d(Tz, hz) + a_4\, d(z, Tfp) + a_5\, d(Tfp, z).\\
&\leq (a_1 + a_4 + a_5)\, d(Tfp, z).
\end{aligned}$$

Since, $hz = z$ we get, $d(z, Tfp) = 0$. So $Tfp = z = Tz$.
Since T is one-one we get, $gp = fp = z$.
Hence z is the unique point of coincidence of (f, g) and by weakly compatibility of (f, g), we get $fz = z = gz$.
Thus we have z is the unique common fixed point of f, g, h and T in X.

Example 2.5. Let $X = R^+ \cup \{0\}$, be a set with the metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y,\\ x + y, & \text{if } x \neq y, \end{cases}$$

then (X, d) is a metric space and f, g, h and $T : X \to X$, defined by

$$f(x) = \begin{cases} 0, & \text{for } x = 0,\\ x + 1, & \text{for } x \neq 0, \end{cases} \quad g(x) = \begin{cases} 0, & \text{for } x = 0,\\ 2x + 1, & \text{for } x \neq 0, \end{cases} \quad T(x) = \begin{cases} 0, & \text{for } x = 0,\\ x + 2, & \text{for } x \neq 0. \end{cases}$$

$$h(x) = x, \ \forall x \in X.$$

Considering the sequence $\{x_n\} = 0, \forall n \in N$, it follows that the pair (f, g) satisfy weakly $CLR^*_{(f,g),T}$, clearly the pairs (f, g) and (h, T) are weakly compatible mappings.

Also, T is one-one. So we have to verify the contraction condition of the above theorem is also satisfied. Here we have the following cases:

$$If(x = y) \Rightarrow \begin{cases} \textbf{Case 1}, & \text{then } x = y = 0,\\ \textbf{Case 2}, & \text{then } x = y > 0, \end{cases} \quad If(x \neq y) \Rightarrow \begin{cases} \textbf{Case 3}, & \text{then } x > y > 0,\\ \textbf{Case 4}, & \text{then } y > x > 0,\\ \textbf{Case 5}, & \text{then } x > y = 0,\\ \textbf{Case 6}, & \text{then } y > x = 0. \end{cases}$$

Then

Case 1: If $x = y = 0$, then the case is trivial as $fx = gx = hx = Tx = 0$.

Case 2:
So, $x = y > 0$. Then we have $Tfx = x + 3, Tgx = 2x + 3, Ty = y + 2 = x + 2, hy = y = x$.

$d(Tfx, hy) = 2x+3$, $d(Tgx, Ty) = 3x+5$, $d(Tfx, Tgx) = 3x+6$, $d(Ty, hy) = 2x+2$, $d(Tgx, hy) = 3x+3$, $d(Tfx, Ty) = 2x+5$. Now by choosing

$$a_1 = a_2 = a_3 = a_4 = a_5 = \frac{1}{6},$$

then $a_1 + a_2 + a_3 + a_4 + a_5 = \frac{5}{6} < 1$, we still have the condition $\sum_{i=1}^{5} a_i < 1$ with each $a_i \in (0,1)$ and also here we get,

$$a_1 d(Tgx, Ty) + a_2 d(Tfx, Tgx) + a_3 d(Ty, hy) + a_4 d(hy, Tgx) + a_5 d(Tfx, Ty) = \frac{1}{6}(13x + 21),$$

and since $d(Tfx, hy) = 2x + 3 = \frac{1}{6}(12x + 18) < \frac{1}{6}(13x + 21) = a_1 d(Tgx, Ty) + a_2 d(Tfx, Tgx) + a_3 d(Ty, hy) + a_4 d(hy, Tgx) + a_5 d(Tfx, Ty)$, that is,

$$d(Tfx, hy) \leq a_1 d(Tgx, Ty) + a_2 d(Tfx, Tgx) + a_3 d(Ty, hy) + a_4 d(hy, Tgx) + a_5 d(Tfx, Ty).$$

Case 3: So, $x > y > 0$. Then we have $Tfx = x + 3, Tgx = 2x + 3, Ty = y + 2, hy = y$.
$d(Tfx, hy) = x+y+3$, $d(Tgx, Ty) = 2x+y+5$, $d(Tfx, Tgx) = 3x+6$, $d(Ty, hy) = 2y+2$, $d(Tgx, hy) = 2x+y+3$, $d(Tfx, Ty) = x + y + 5$. Now by choosing

$$a_1 = a_3 = a_4 = a_5 = \frac{1}{5}, \text{ then } a_1 + a_3 + a_4 + a_5 = \frac{4}{5},$$

so if we take $a_2 = \frac{1}{10}$ we still have the condition $\sum_{i=1}^{5} a_i < 1$ with each $a_i \in (0,1)$ and also here we get,

$$a_1 d(Tgx, Ty) + a_3 d(Ty, hy) + a_4 d(hy, Tgx) + a_5 d(Tfx, Ty) = x + y + 3 = d(Tfx, hy),$$

and since $a_2 d(Tfx, Tgx) \geq 0$, we get,

$$d(Tfx, hy) \leq a_1 d(Tgx, Ty) + a_2 d(Tfx, Tgx) + a_3 d(Ty, hy) + a_4 d(hy, Tgx) + a_5 d(Tfx, Ty).$$

Case 4: Follows from **Case 3**.

Case 5: If $x > y = 0$. Then we have $Tfx = x + 3, Tgx = 2x + 3, Ty = 0, hy = 0$.
$d(Tfx, hy) = x + 3$, $d(Tgx, Ty) = 2x + 3$, $d(Tfx, Tgx) = 3x + 6 = 2(x + 3)$, $d(Ty, hy) = 0$, $d(Tgx, hy) = 2x + 3$, $d(Tfx, Ty) = x + 3$. Now by choosing

$$a_2 = \frac{1}{2} \text{ and } a_1 + a_3 + a_4 + a_5 < \frac{1}{2},$$

we still have the condition $\sum_{i=1}^{5} a_i < 1$ and also here we get, $x + 3 = a_2 \; 2(x + 3)$ that is we have

$$d(Tfx, hy) = a_2 d(Tfx, Tgx),$$

and

$$a_1 d(Tgx, Ty) + a_3 d(Ty, hy) + a_4 d(hy, Tgx) + a_5 d(Tfx, Ty) \geq 0.$$

therefore we get,

$$d(Tfx, hy) \leq a_1 d(Tgx, Ty) + a_2 d(Tfx, Tgx) + a_3 d(Ty, hy) + a_4 d(hy, Tgx) + a_5 d(Tfx, Ty).$$

Case 6: Follows from **Case 5**.
Thus using the above Theorem 2.4 we get 0 is the unique common fixed point of f, g, h and T in X.

Theorem 2.6. Let (X, d) be a metric space and $f, g, h, T : X \to X$ such that (f, g) satisfy weakly $CLR^*_{(f,g),T}$ property and for each $x, y \in X$,

$$d(Tfx, hy) \leq \lambda \ max \ m_1(x, y).$$

where, $\lambda \in (0, 1)$. Then if T is one-one function and (f, g) and (h, T) are weakly compatible mappings then f, g, h and T have a unique common fixed point in X.

Proof : Since f, g and $T : X \to X$ such that (f, g) satisfy weakly $CLR^*_{(f,g),T}$ property. There exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tgx_n = z$, for some $z \in g(X) \cap T(X)$. Let $z = T(u)$, for some $u \in X$.

Suppose that $z \neq Tz$ or $z \neq Tgu$.

Taking $x = x_n$ and $y = u$ in the theorem we get,

$$d(Tfx_n, hu) \leq \lambda \ max \ m_1(x_n, u), \ where \ \lambda \in (0, 1).$$

$m_1(x_n, u) = \{d(Tgx_n, Tu), d(Tfx_n, Tgx_n), d(hu, Tu), \frac{d(Tgx_n,hu)+d(Tu,Tfx_n)}{2}, \frac{d(Tfx_n,Tgx_n)d(hu,Tu)}{1+d(Tgx_n,Tu)},$

$\frac{d(Tfx_n,Tu)d(hu,Tgx_n)}{1+d(Tgx_n,Tu)}, d(Tfx_n, Tgx_n)\frac{1+d(Tgx_n,hu)+d(Tu,Tfx_n)}{1+d(Tfx_n,Tgx_n)+d(Tu,hu)}\}.$

Let $n \to \infty$ then we get,

$$\lim_{n\to\infty} d(Tfx_n, hu) \leq \lambda \ max \ \lim_{n\to\infty} \ m_1(x_n, u) = \lambda d(hu, z).$$

where, $\lambda \in (0, 1)$. Thus we have,

$$d(hu, z) \leq \lambda d(hu, z).$$

where, $\lambda \in (0, 1)$. So we get $hu = z$, which gives $hu = z = Tu$.

Now suppose $w \neq u$ such that $hw = Tw$. Then Taking $x = x_n$ and $y = w$ in the theorem we get $hw = z$, which gives $hw = z = Tw$. Thus z is the unique point of coincidence. By weakly compatibility of (h, T) it follows that z is the unique common fixed point theorem of h, T.

Proceeding as in the proof of the Theorem 2.4, we get the desired result.

Theorem 2.7. Let (X, d) be a metric space and $f, g, h, T : X \to X$ such that (f, g) satisfy weakly $CLR^*_{(f,g),T}$ property and for each $x, y \in X$,

$$d(Tfx, hy) \leq \lambda \ max \ m_i(x, y).$$

where, $\lambda \in (0, 1)$ and $i = 2, 3, 4$. Then if T is one-one function and (f, g) and (h, T) are weakly compatible mappings then f, g, h and T have a unique common fixed point in X.

Proof : The proof is same as the above Theorem 2.6.

Theorem 2.8. Let (X, d) be a metric space and $f, g, h, T : X \to X$ such that (f, g) satisfy weakly $CLR^*_{(f,g),T}$ property and for each $x, y \in X$,

$$d(Tfx, hy) \leq \lambda max\{d(Tgx, Ty), d(Tfx, Tgx), d(Ty, hy), d(hy, Tgx), d(Tfx, Ty)\}.$$

where, $\lambda \in (0, 1)$. Then if T is one-one function and (f, g) and (h, T) is weakly compatible mappings then f, g, h and T have a unique common fixed point in X.

Proof : The proof is same as the above Theorem 2.4.

Theorem 2.9. Let (X, d) be a metric space and f, g, h, and T be self mappings of X satisfying the inequality

$$F(\psi(d(Tfx, hy)), \psi(d(Tgx, Ty)), \psi(d(Tgx, Tfx)), \psi(d(Ty, hy)), \psi(d(Tgx, hy)), \psi(d(Ty, Tfx))) \leq 0,$$

for all $x, y \in X$, $F \in \mathcal{F}$ and ψ is an altering distance.
If f, g, and T satisfy weakly $CLR^*_{(f,g),T}$ - property and T is one-one, then $C(f, g) \neq \phi$, *and* $C(h, T) \neq \phi$. Moreover, if (f, g) and (h, T) are weakly Compatible, then f, g, h, and T have a unique common fixed point.

Proof: Since f, g, and T satisfy weakly $CLR^*_{(f,g),T}$ - property, there exists a sequences x_n in X

$$\lim_{n\to\infty} Tfx_n = \lim_{n\to\infty} Tgx_n = z,$$

for some $z \in g(X) \cap T(X)$.
Hence $z \in T(X)$, which implies $z = Tu$ for some $u \in X$.
By taking $x = x_n$ and $y = u$ we have,

$$F(\psi(d(Tfx_n, hu)), \psi(d(Tgx_n, Tu)), \psi(d(Tgx_n, Tfx_n)), \psi(d(Tu, hu)), \psi(d(Tgx_n, hu)), \psi(d(Tu, Tfx_n))) \leq 0,$$

Letting $n \to \infty$, we obtain

$$F(\psi(d(z, hu)), \psi(d(z, Tu)), \psi(d(z, z)), \psi(d(z, hu)), \psi(d(z, hu)), \psi(d(z, z))) \leq 0,$$
$$F(\psi(d(z, hu)), 0, 0, \psi(d(z, hu)), \psi(d(z, hu)), 0) \leq 0,$$

a contradiction to $(F2)$ if $d(z, hu) > 0$. Hence, $d(z, hu) = 0$, which implies $z = hu = Tu$ and $C(h, T) \neq \phi$. If suppose that w is another point of coincidence of (h, T) then $w = hu = Tu$. Then we get, $Tz = Thu = Tw$ and since T is one-one we get $z = w$. Therefore the point of coincidence of (h, T) is unique and since (h, T) is weakly compatible mappings it follows that z is the unique common fixed point of (h, T) so we have, $z = hz = Tz$.

Again $z \in g(X)$, which implies $z = gv$ for some $v \in X$. By taking $x = v$ and $y = u$ we get,

$$F(\psi(d(Tfv, hu)), \psi(d(Tgv, Tu)), \psi(d(Tgv, Tfv)), \psi(d(Tu, hu)), \psi(d(Tgv, hu)), \psi(d(Tu, Tfv))) \leq 0,$$

Letting $n \to \infty$, we obtain

$$F(\psi(d(Tfv, z)), \psi(d(Tz, Tu)), \psi(d(z, Tfv)), \psi(d(z, z)), \psi(d(z, z)), \psi(d(z, Tfv))) \leq 0,$$

$$F(\psi(d(Tfv, z)), 0, \psi(d(z, Tfv)), 0, 0, \psi(d(z, Tfv))) \leq 0,$$

a contradiction to $(F1)$ if $d(z, Tfv) > 0$.

Hence, $d(z, Tfv) = 0$, which implies $z = Tfv$ so $Tz = Tfv$ and T is one-one which implies that $z = fv = gv$ and $C(f, g) \neq \phi$.

Similarly, we can show that the point of coincidence is unique. So z is a unique common fixed point of (f, g). Thus we have $z = hz = Tz = fz = gz$.

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