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More on P-closed spaces

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Abstract

In [1] the following problems were listed as open: Problem 14. Is a regular space in which every closed subset is regular-closed compact? Problem 15. Is a Urysohn-space in which every closed subset is Urysohn-closed compact? To answer the question for Hausdorff-closed spaces in the affirmative, M. H. Stone [12] used Boolean rings and M. Katětov [10] used topological methods. In this article, all three questions are answered affirmatively using filters.

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Rigid, Ultrafilters, P-closed.
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1. Introduction

In this article it is proved that if every closed subset of a P-space is P-closed then the space is compact, where the property P is Hausdorff, Urysohn or regular.

Recently, the authors have published a series of articles, [3], [4], [5], [9], to mention some of them, addressing these and other related problems. These problems were also looked at in [11]. We adopted a general approach in our articles to prove compactness using the result that a space is compact if and only if, every ultrafilter converges. In this article, a filter consisting of open sets, was constructed from an ultrafilter such that each member of the filter contains a member of the ultrafilter. This open filter is contained in an open ultraliter [13, page 83]. It is proved then that, if the original ultrafilter does not contain a single point set, the following result holds:

Suppose that X is a Hausdorff (Urysohn) [regular] space which is H-closed (Urysohn-closed) [regular-closed] with the property that every closed subset of X is H-closed (Urysohn-closed) [regular-closed]. If every open ultrafilter converges in X, then every ultrafilter converges in X.

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Also, a second proof for compactness for such spaces is provided using open covers and the above result as well. The results that a Hausdorff (Urysohn) [regular] space is H-closed (Urysohn-closed) [regular-closed] if every open cover of the space has a finite subfamily the closures (u-closures) [s-closures] of members of which cover the space, are used to give the second proof of the Main Theorem.

Throughout this paper all spaces are Hausdorff. For a subset A of X, clA represents the closure of A; $\Sigma(x)$ represents the collecton of open sets V with x in V; and $\Gamma(x)$ represents the collection of open sets containing a closed neighborhood of x. The adherence of a filterbase $\mathcal{F}$, denoted as $adh\mathcal{F}, = \cap_{F\in\mathcal{F}} clF$.

An open filterbase $\mathcal{F}$ is a Urysohn filterbase [1] if and only if for each $p \notin adh\mathcal{F}$, there is an open set U containing p and $V \in \mathcal{F}$ such that $clU \cap clV = \emptyset$. An open filterbase is a regular filterbase [2] if and only if for each $U \in \mathcal{F}$ there is a $V \in \mathcal{F}$ such that $clV \subseteq U$. In [6], it is established that a point x is in the u-closure of $A \subset X$, denoted as $x \in cl_u(A)$, if $V \cap A \neq \emptyset$ for each $V \in \Gamma(x)$, if and only if $cl_u(V) \cap A \neq \emptyset$ for every $V \in \sum(x)$. Herrington [7] defined the concept of u-adherence of a filterbase. A point $x \in X$ is called a u-adherent point of a filterbase $\mathcal{F}$, denoted as $x \in adh_u\mathcal{F}$, if for each $F \in \mathcal{F}$ and $U \in \Gamma(x)$, $clU \cap F \neq \emptyset$. That is, $adh_u\mathcal{F} = \cap_{F\in\mathcal{F}} cl_uF$. Herrington [7] showed that a Urysohn space is Urysohn-closed if and only if each filterbase $\mathcal{F}$ on the space satisfies $adh_u\mathcal{F} \neq \emptyset$.

In [8] Herrington called a family of open sets, $\mathcal{G}$, a shrinkable family of open sets about a point $x \in X$ if for each $U \in \mathcal{G}$, there is a $V \in \mathcal{G}$ such that $x \in U \subseteq clU \subseteq V$. A point x *is in the s-closure of* $A \subset X$, *denoted as* $x \in cl_s(A)$, *if and only if* $V \cap A$ $(cl_s(V) \cap A) \neq \emptyset$ *for every* $V \in \mathcal{G}(x)$ $(V \in \sum(x))$, *where* $\mathcal{G}(x)$ *is a shrinkable family of open sets around* x. Herrington [7] defined a point $x \in X$ to be in the s-adherence of a filterbase $\mathcal{F}$, denoted as $x \in adh_s\mathcal{F}$, if for each shrinkable family $\mathcal{G}$ of open sets about x and $F \in \mathcal{F}$, there is a $V \in \mathcal{G}$ such that $F \cap V \neq \emptyset$. That is, $adh_s\mathcal{F} = \bigcap_{\mathcal{F}} cl_s(F)$ [8]. It was then proved that a regular space is regular-closed if and only if each filterbase on the space has non-empty s-adherence.

2. Main Results

It is clear that in a Urysohn (regular) space an open ultrafilter has a single point of u-adherence (s-adherence) and also a single point of θ adherence, if there is any. Hence, in a Hausdorff-closed (Urysohn-closed) [regular-closed] space, every open ultrafilter converges to this single adherent point.

If $\mathcal{W}$ is an ultrafilter on a space X and $\mathcal{C} = \{C \text{ open in X } : W \subseteq C \text{ for some } W \in \mathcal{W}\}$, then $\mathcal{C} = \{C \in \mathcal{W} : C \text{ is open}\}$. Moreover, $\mathcal{C}$ is an open filter on X. Then, there is an open ultrafilter $\mathcal{U}$ containing $\mathcal{C}$ [13].

Theorem. If X is a Hausdorff (Urysohn) [regular] space which is H-closed (Urysohn-closed) [regular-closed] with the property that every closed subset of X is H-closed (Urysohn-closed) [regular-closed], then the space is compact.

Proof 1. Let $\mathcal{W}$ be an ultrafilter on X. If $\cap_{W\in\mathcal{W}} W = \{x\}$ for some $x \in X$, then $\mathcal{W} \to x \in X$, as then x will be the single point of adherence, (u-adherence) [s-adherence], since the singleton set $\{x\}$ is a member of $\mathcal{W}$.

Now, suppose $\bigcap_{W\in\mathcal{W}} W$ is not a single point set and let $\mathcal{O}$ be the open members of $\mathcal{W}$. Then $\mathcal{O}$ is an open filter, and there is an open ultrafilter $\mathcal{U}$ containing $\mathcal{O}$ on X and has a non-empty adhernce (u-adherence) [s-adherence], say $x \in X$. Hence $\mathcal{U} \to x$ $(\to_u x)[\to_s x]$, having a single point of θ adherence (u-adherence) [s-adherence]. Let V be an open set such that $x \in V$. Then $X - V$ is a closed subset of X and hence is H-closed (Urysohn-closed) [regular-closed]. Suppose that $\mathcal{W} \not\to x$. So, for each $W \in \mathcal{W}$, $W \cap (X - V) \neq \emptyset$. Since each $O \in \mathcal{O}$ is a member of $\mathcal{W}$, for each $O \in \mathcal{O}$, $O \cap (X - V) \neq \emptyset$. Then $\mathcal{F} = \{B \in \mathcal{U} : B \cap (X - V) \neq \emptyset\}$ is an open filter on the closed set $X - V$ which has a non-empty adherence (u-adherence) [s-adherence] on $X - V$. This is a contradiction, since $\mathcal{U} \to x$ $(\to_u x)[\to_s x]$, hence $\mathcal{U}$ has adherence (u-adherence) [s-adherence] in V and in $(X - V)$. Hence $\mathcal{W} \to x \in X$. Thus the proof is complete.

In the proof of the Theorem, we used the argument which proves the following:

Corollary. Suppose X is a Hausdorff (Urysohn) [regular] space which is H-closed (Urysohn-closed) [regular-closed] with the property that every closed subset of X is H-closed (Urysohn-closed) [regular-closed]. If every open ultrafilter converges in X, then every ultrafilter converges in X.

Below, the above Corollary, and results that a Hausdorff (Urysohn) [regular] space is H-closed (Urysohn-closed) [regular-closed] if every open cover of the space has a finite subfamily the closures (u-closures) [s-closures] of members of which cover the space, are used to give another proof of the Main Theorem.

Proof 2. Let $\mathcal{U}$ be an open ultrafilter on X. Suppose that $\mathcal{U}$ does not converge in X. Then $adh\mathcal{U} = \emptyset$. So, for each $x \in X$, there is an open set V_x containing x and a $U_x \in \mathcal{U}$ such that $U_x \cap V_x = \emptyset$. The family $\mathcal{V} = \{V_x | x \in X\}$ is an open cover of the H-closed (Urysohn-closed) [regular-closed] space X and hence has a finite subfamily $\{V_{x_i}, i = 1, 2, \ldots, n\}$ such that $X = \cup_{i=1}^n clV_{x_i}(X = \cup_{i=1}^n cl_u V_{x_i})[X = \cup_{i=1}^n cl_s V_{x_i}]$. Let $U = \cap_{i=1}^n U_{x_i}$. Then $U \in \mathcal{U}$ and $U \cap (\cup_{i=1}^n clV_{x_i})(U \cap (\cup_{i=1}^n cl_u V_{x_i}))[U \cap (\cup_{i=1}^n cl_s V_{x_i})] = \emptyset$. However, each $clV_{x_i}(cl_u V_{x_i})[cl_s V_{x_i}]$ is a closed set and hence is H-closed, (Urysohn-closed) [regular-closed]. Since $\mathcal{U}$ is a filter on X, there is an $x \in X$ such that $\mathcal{F} = \{F \cap clV_{x-i}(F \cap cl_u V_{x_i})[F \cap cl_s V_{x_i}], F \in \mathcal{U}\}$ is an open filter on $clV_{x-i}(cl_u V_{x-i})[cl_s V_{x-i}]$ and hence has non-empty adherence (u-adherence) [s-adherence] . This is a contradiction since $(U \cap clV_{x-i})((U \cap cl_u V_{x-i}))[U \cap cl_s V_{x_i})] = \emptyset$ for each $i \in \{1, 2, \ldots, n\}$. Hence $\mathcal{U}$ has non-empty adherence and hence $\mathcal{U}$ converges. Thus, in view of the Corollary, every ultrafilter on X converges and thus X is compact.

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